Extra homework

Discrete mathematics I

- Not required, only for plus points (max. 10 points in total).
- This file is updated after every class.
- See the website for more information.

Class 1 (10 Feb) – deadline: **3 Mar**

1. Consider the following predicates: $P(x)$: x is a prime number, $E(x)$: x is even (where x is a positive integer). Translate the following formula into an English sentence as simple as possible, using *at most 12 words*:
$$\exists x(E(x) \wedge P(x)) \wedge \neg\exists x(E(x) \wedge P(x) \wedge \exists y(\neg(x = y) \wedge E(y) \wedge P(y))).$$
Is is true or false? Why? **(1 point)**

2. (SAT-problem) Let $x_1, x_2, \ldots, x_n$ be boolean variables (*true* or *false*), and consider the formulas that can be formed using these variables and the logical operations $\wedge$, $\vee$ and $\neg$. A formula is *satisfiable* if the values of the variables can be chosen so that the formula becomes *true*. For example, $F(x_1, x_2) = x_1 \wedge x_2$ is satisfiable, since if $x_1 = x_2 = \mathit{true}$, then F is also *true*. But $G(x_1) = x_1 \wedge \neg x_1$ is not satisfiable, because it is *false* for either value of x_1. Write a program that decides for any given n-variable formula whether it is satisfiable. What is the largest n for which the program runs within a reasonable time limit (e.g. in 1 hour)? **(2 points)**

Class 2 (17 Feb) – deadline: **10 Mar**

3. Natural numbers can be built using only set theory by the following recursion:
   - $0 := \emptyset$,
   - $\forall n \in \mathbb{N} : n + 1 := n \cup \{n\}$.

   For example, $1 = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}$. Write down 2, 3 and 4 in their set representation. **(1 point)**

Class 3 (24 Feb) – deadline: **17 Mar**

4. Is the following statement true for every proposition A, B, C and D?
$$((A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (C \Rightarrow D)) \iff ((\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge D) \vee (B \wedge C \wedge D))$$
(1 point)

Class 4 (3 Mar) – deadline: **24 Mar**

5. From another exercise (PS2/Q15/b), we have that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$, i.e. the symmetric difference is associative. Therefore, we can omit the parentheses and simply write $A \triangle B \triangle C$. Prove that $A \triangle (B \triangle C \triangle D) = (A \triangle B \triangle C) \triangle D$. **(1 point)**

6. Let A and B be sets such that $|A| = m$ and $|B| = n$ ($m, n \in \mathbb{N}$). How many different binary relations exist from A to B? **(1 point)**

Class 5 (10 Mar) – deadline: **31 Mar**

7. Let $R \subseteq X \times X$ be a binary relation. Prove that $R = R^{-1} \iff R \subseteq R^{-1}$. **(1 point)**

8. Write a program that takes two finite binary relations R and S, and calculates $R \circ S$ and $S \circ R$. **(1 point)**

Class 6 (17 Mar) – deadline: **7 Apr**

9. Let $R, S \subseteq X \times X$ be symmetric relations. Prove that $R \circ S$ is symmetric if and only if $R \circ S = S \circ R$. **(1 point)**

10. How many different
    a) reflexive, b) irreflexive, c) symmetric, d) antisymmetric
    relations exist on a given n-element set? **(2 points: 0.5 point each)**

Class 7 (24 Mar) – deadline: **14 Apr**

11. Prove that the inverse of a partial order is also a partial order. **(1 point)**

12. Prove that every finite partially ordered set contains at least one maximal and one minimal element. **(1 point)**

13. Let $m \in \mathbb{R}^+$ fixed, $A = \{\text{all isosceles triangles with height } m\}$ and $B = \mathbb{R}^+$.
    Let $R = \{(a, b) \in A \times B \mid \text{area of } a \text{ equals } b\}$.
    Prove that R is a function, and decide whether it is injective, surjective, and/or bijective. **(1 point)**

Class 8 (31 Mar) – deadline: **28 Apr**

14. Prove that the multiplication of complex numbers is distributive over addition, i.e.:
    $\forall x, y, z \in \mathbb{C} : x(y + z) = xy + xz.$ **(1 point)**

15. Using only the definition of multiplication of complex numbers (!), prove that every $z \in \mathbb{C} \setminus \{0\}$ has a multiplicative inverse, i.e. $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$ and $z^{-1}z = 1$. **(1 point)**

Class 9 (7 Apr) – deadline: **28 Apr**

16. The number of permutations of $n + 2$ distinct elements is 20 times more than the number of permutations of n distinct elements. What is n? **(1 point)**

17. Write a program that takes a finite set A (where $|A| \leq 6$) and prints all permutations of A. **(1 point)**

Class 10 (14 Apr) – deadline: **5 May**

18. In how many ways can we split 1000000 into the product of three natural numbers if the order of the factors a) matters, b) doesn't matter? **(1 point)**

19. Write a program that takes a finite set A (where $|A| \leq 6$) and an integer k (where $1 \leq k \leq 5$), and it prints all combinations of A without repetition. **(1 point)**

Class 11 (28 Apr) – deadline: **19 May**

20. Solve the equation $0.7 \cdot \binom{25}{x} = \binom{23}{x}$ on the set $\{0, 1, 2, \ldots, 23\}$. **(1 point)**

21. Consider all strings of length six composed of the digits 0-9 (any number of times, including zero times). How many of these strings are such that they do not contain the substring 42 (i.e. the digits 4 and 2 in this order, next to each other)? **(1 point)**

Class 12 (5 May) – deadline: **26 May**

22. Prove that if 33 points are selected inside a square so that no 3 points lie on the same line, then there must be 3 points among the 33 such that the area of the triangle determined by these points is at most 1/32 of the area of the whole square. **(2 points)**

Class 13 (12 May) – deadline: **28 May**

23. Prove that any simple graph with $n \geq 2$ vertices contains two vertices with the same degree. **(1 point)**

24. Let G be a simple graph with 6 vertices. Prove that either G or $\overline{G}$ contains a 3-vertex complete graph (K_3) as a subgraph. **(1 point)**