

Problem set 7: Permutations, variations, combinations

Summary of theory

permutations (without repetition): A *permutation* (without repetition) of a finite set A is a sequence containing each element of A exactly once. (In other words, a possible ordering of the elements of A .)

The number of permutations of a finite set: The number of permutations of an n -element set is $P_n = n!$. (By definition, for every $n \in \mathbb{N}^+$: $n! = n(n-1)(n-2) \cdot \dots \cdot 2 \cdot 1$, and $0! = 1$.)

permutations with repetition: Let $a_1, a_2, \dots, a_m$ be m different objects and $k_1, k_2, \dots, k_m \in \mathbb{N}$. A sequence of length $n = k_1 + k_2 + \dots + k_m$ containing each a_i exactly k_i times ($1 \leq i \leq m$) is called a *permutation with repetition* of $a_1, a_2, \dots, a_m$, with repetition numbers $k_1, k_2, \dots, k_m$.

The number of permutations with repetition: The number of permutations with repetition of m different objects with repetition numbers $k_1, k_2, \dots, k_m$ is: $\frac{n!}{k_1! \cdot k_2! \cdot \dots \cdot k_m!}$, where $n = k_1 + k_2 + \dots + k_m$.

variations (without repetition) (or partial permutations): Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by some elements of A containing each element of A at most once, is called a *k-variation without repetition* of A .

The number of variations: Let $k \in \mathbb{N}$. The number of k -variations without repetition of an n -element set is: $V_n^k = P_n^k = P(n, k) = n \cdot (n-1) \cdot \dots \cdot (n-k+1) = n!/(n-k)!$ if $k \leq n$ and is 0 if $k > n$.

variations with repetition: Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by some elements of A (any element may occur more than once), is called a *k-variation with repetition* of A .

The number of variations with repetition: Let $n, k \in \mathbb{N}$. The number of k -variations with repetition of an n -element set is: n^k .

combinations (without repetition): Let $k \in \mathbb{N}$. A k -element subset of a set A is called a *k-combination* of A .

The number of combinations: Let $n, k \in \mathbb{N}$. The number of k -combinations of an n -element set is: $C_n^k = C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ if $k \leq n$ and is 0 if $k > n$.

combinations with repetition: Let $k \in \mathbb{N}$. A *k-combination with repetition* (or *k-multiset*) from a set A is a selection of k (not necessarily distinct) elements from A , where repetition is allowed and the order does not matter.

The number of combinations with repetition: Let $n, k \in \mathbb{N}$. The number of k -combinations with repetition of an n -element set is: $\binom{n+k-1}{k}$.

Questions

Question 1: In how many ways can you arrange 1, 2, 3 or 5 distinct characters, respectively, into order?

Question 2:

- (a) At a literary event 5 different poems are presented. In how many different orders can the poems be read out?
- (b) In how many different orders can 6 people be seated along a bench?

- (c) 12 students have agreed to have a meeting. In how many different orders can they arrive at the meeting if we assume that they all arrive at different times?
- (d) How does the answer to question (b) change if instead of a bench, the 6 people are seated around a round table?

Question 3: In how many different ways can you arrange into order

- (a) 3 red, 1 blue and 1 white
(b) 3 red, 2 blue and 1 white

balls if we do not distinguish between balls of the same colour?

Question 4: A box contains 16 balls: 10 white, 4 red and 2 blue balls. We take the balls out of the box one-by-one. In how many different orders can the balls be removed from the box, if we do not distinguish between balls of the same colour?

Question 5: How many different 5-digit numbers can be formed using exactly the digits

- (a) 1, 2, 3, 4, 5?
(b) 1, 1, 2, 3, 4?
(c) 1, 1, 2, 2, 2?

(Each digit has to be used exactly as many times as many times it appears in the list.)

Question 6: 15 students are taking part in a running race. How many different outcomes are possible for the first three places, if we assume that there are no equal finishes?

Question 7: In how many different ways can we distribute 6 different books among 10 pupils if everyone can get *at most* one book?

Question 8: How many different 5-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, 7, 8 if

- (a) each digit can be used *at most* once?
(b) any digit can be used more than once?

Question 9: How many 6-digit numbers exist in which all digits are different (i.e. which do not contain repeated digits) in the (a) base-10 (b) base-8 (c) base-12 number system?

Question 10: How many different outcomes are possible when

- (a) flipping a coin 10 times, (b) rolling a die 10 times,
if the order of the results matters?

Question 11: A multiple choice test consists of 30 questions. For each question 5 possible answers are provided, out of which exactly one answer needs to be selected. In how many different ways can the test be completed?

Question 12: In how many different ways can we distribute 6 identical books among 20 students, if each student can be given *at most* one book?

Question 13: In how many different ways can 4 cards be handed out to a player from a deck of 32 cards? (It does not matter, what order the 4 cards are handed out.)

Question 14: A lottery ticket contains the numbers $1, 2, 3, \dots, 90$. When filling in the ticket you need to select and mark 5 numbers from among these 90 numbers. (You are trying to guess which 5 numbers will be the winning numbers drawn later.) In how many different ways can you fill in the lottery ticket?

Question 15: Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- (a) How many 3-element subsets does A have?
- (b) How many 5-element subsets does A have which contains 7?
- (c) How many 4-element subsets does A have which contains only odd numbers?
- (d) How many subsets does A have in total?

Question 16: We draw 6 cards from a deck of 32 cards (without replacement). How many different outcomes are possible

- (a) if the order in which the cards are drawn matters?
- (b) if the order in which the cards are drawn does not matter?

Question 17: In how many different ways can we distribute 4 apples among 28 children, if any child can receive more than one apple?

Question 18: In a post office 12 types of cards are sold. In how many different ways can we purchase 5 cards (we assume that the post office has at least 5 copies of each card in stock)?

Question 19: In how many different orders can 4 couples sit along a bench if each person would like to sit next to his/her partner?

Question 20: A company of 8 people would like to sit down at a round table. In how many different orders can they sit around the table, if two particular members of the group: Anna and Ignatious would like to sit next to each other?

Question 21: Given that the number of permutations of $n + 2$ (distinct) element equals 20 times the number of permutations of n (distinct) elements, find the value of n .

Further questions

Question 22: Write a program that takes as input a finite set A with $|A| \leq 6$ and prints all permutations without repetition of A .

Question 23: Write a program that takes as input a finite set A with $|A| \leq 6$ and an integer $1 \leq k \leq |A|$, and prints all

- (a) k -variations without repetition of A ;
- (b) k -combinations without repetition of A .

Question 24: Write a program that takes as input a finite set A with $|A| \leq 5$ and an integer $1 \leq k \leq 4$, and prints all k -variations with repetition of A .