

Problem set 7:

Complex numbers – algebraic and polar forms of complex numbers

Summary of theory

Algebraic form and basic concepts of complex numbers

complex number, algebraic form: Expressions of the form $a+bi$ where $a, b \in \mathbb{R}$ are called *complex numbers*, where on the set of all complex numbers $\mathbb{C}$ addition (+) and multiplication ($\cdot$) are defined as follows:
 $\forall a+bi, c+di \in \mathbb{C}$:

- $(a+bi) + (c+di) = a+c + (b+d)i$
- $(a+bi) \cdot (c+di) = ac - bd + (ad+bc)i$

The above form $a+bi$ ($a, b \in \mathbb{R}$) is called the *algebraic form* of complex numbers.

real part, imaginary part, conjugate, absolute value: Let $z = a+bi$, where $a, b \in \mathbb{R}$. The *real part* of z is $\text{Re}(z) = a$, the *imaginary part* of z is $\text{Im}(z) = b$, the *conjugate* of z is $\bar{z} = a-bi$ and the *absolute value* of z is $|z| = \sqrt{a^2+b^2}$.

A method of finding the quotient of two complex numbers in algebraic form: Let $a+bi, c+di \in \mathbb{C}$ be complex numbers, where $c+di \neq 0$. Then the quotient $\frac{a+bi}{c+di}$ can be found in algebraic form by ‘expanding the fraction by the conjugate of the denominator’ as follows:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i$$

Polar form of complex numbers

polar form: The *polar form* of a complex number $z \neq 0$ is $z = r(\cos \varphi + i \cdot \sin \varphi)$, where $r = |z|$.

Note: In the polar form φ is the angle from the positive real axis to the position vector of z in the complex plane. This angle is not unique: $r(\cos \varphi + i \cdot \sin \varphi) = r(\cos(\varphi + 2k\pi) + i \cdot \sin(\varphi + 2k\pi))$ for every $k \in \mathbb{Z}$.

A method for converting complex numbers from algebraic to polar form: For any $z = a+bi$ ($a, b \in \mathbb{R}$):

$$r = |z| = \sqrt{a^2+b^2} \text{ and } \varphi = \begin{cases} \cos^{-1} \frac{a}{|z|} & \text{if } b \geq 0 \\ -\cos^{-1} \frac{a}{|z|} & \text{if } b < 0 \end{cases}$$

Multiplying, dividing and raising complex numbers to the n^{th} power in polar form: For every complex number $z = |z|(\cos \varphi + i \cdot \sin \varphi)$ and $w = |w|(\cos \psi + i \cdot \sin \psi)$ and every $n \in \mathbb{N}^+$:

1. $zw = |z||w|(\cos(\varphi + \psi) + i \cdot \sin(\varphi + \psi))$
2. $\frac{z}{w} = \frac{|z|}{|w|}(\cos(\varphi - \psi) + i \cdot \sin(\varphi - \psi))$
3. $z^n = |z|^n(\cos(n\varphi) + i \cdot \sin(n\varphi))$ (De Moivre’s formula)

n^{th} roots of a complex number: Let $z \in \mathbb{C}$ and $n \in \mathbb{N}^+$. Then $w \in \mathbb{C}$ is an n^{th} root of z if $w^n = z$.

Formula for the n^{th} roots of a complex number: Let $z = |z|(\cos \varphi + i \cdot \sin \varphi)$ be an arbitrary nonzero complex number and $n \in \mathbb{N}^+$. Then the n^{th} roots of z are:

$$w_k = \sqrt[n]{|z|} \left(\cos \left(\frac{\varphi}{n} + \frac{2k\pi}{n} \right) + i \cdot \sin \left(\frac{\varphi}{n} + \frac{2k\pi}{n} \right) \right), k = 0, 1, \dots, n-1.$$

Questions

Algebraic form of complex numbers

Q1 Let $z \in \mathbb{C}$, $z = -2 + 7i$. Find the following:

- (a) $\operatorname{Re} z$ (b) $\operatorname{Im} z$ (c) $-z$ (d) $\bar{z}$ (e) $|z|$

Q2 Calculate the following, giving your answers in algebraic form.

- (a) $\sqrt{-16}$ (b) $\sqrt{-25}$ (c) $(2i)^2$ (d) $2i + 5i$ (e) $\frac{4i}{2i}$

Q3 Calculate the value $\frac{4+3i}{(2-i)^2}$ giving your answer in algebraic form.

Q4 Solve the following equation on the set of complex numbers: $\frac{z+i-3i\bar{z}}{z-4} = i-1$.

Q5 Find the complex number(s) $z \in \mathbb{C}$ satisfying the conditions:

$$\left| \frac{z-3}{2-\bar{z}} \right| = 1 \wedge \operatorname{Re} \left(\frac{z}{2+i} \right) = 2$$

Polar form of complex numbers

Q6 Let $z \in \mathbb{C}$, $z = 2 + 5i$. Find the absolute value and the argument of z . Represent z on the complex plane (also called Gaussian plane).

Q7 Write the following complex numbers in polar form:

- (a) $1+i$ (e) $4i$
 (b) $-\sqrt{3}+i$ (f) i
 (c) $\frac{9}{2} - \frac{9\sqrt{3}}{2}i$ (g) 10
 (d) $-\frac{\sqrt{14}}{2} - \frac{\sqrt{14}}{2}i$

Q8 Calculate the following, using the polar form of complex numbers:

$$(a) \left(\frac{9}{2} - \frac{9\sqrt{3}}{2}i \right) \left(-\frac{\sqrt{14}}{2} - \frac{\sqrt{14}}{2}i \right)$$

$$(b) \left(-\frac{3\sqrt{3}}{2} - \frac{3}{2}i \right) \left(\frac{\sqrt{3}}{3} + \frac{1}{3}i \right)$$

$$(c) \frac{-\frac{3\sqrt{3}}{2} - \frac{3}{2}i}{\frac{\sqrt{3}}{3} + \frac{1}{3}i}$$

$$(d) \left(\frac{5\sqrt{3}}{12} - \frac{5}{12}i \right)^{10}$$

$$(e) \left(-\frac{\sqrt{10}}{2} - \frac{\sqrt{10}}{2}i \right)^{15}$$

$$(f) \left(\frac{5}{2} - \frac{5\sqrt{3}}{2}i \right)^{23}$$

$$(g) (1+i)^8 \cdot (5\sqrt{3} - 5i)^3$$

$$(h) \left(\frac{\frac{3}{2} + \frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2} + \frac{5}{2}i} \right)^{12}$$

$$(i) \left(1 - \frac{\sqrt{3} - i}{2} \right)^{24}$$

Q9 Express $z = \frac{(1+i)^8}{(1-\sqrt{3}i)^6}$ in algebraic form.

Q10 Determine the complex roots below:

(a) 2nd roots of -60 ;

(b) 3rd roots of -60 ;

(c) 6th roots of $1 - \sqrt{3}i$;

(d) 5th roots of $-7\sqrt{3} + 7i$;

(e) 8th roots of $-\frac{7}{2} + \frac{7}{2}i$;

(f) 2nd roots of $-6\sqrt{3} + 6i$;

(g) 7th roots of $\frac{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^8}{(1+i)^5}$;

Q11 Using the polar form of complex numbers, calculate the value of

$$z = \frac{(2 + 2\sqrt{3}i)^{10}}{(-1 + i)^{83}},$$

giving your answer both in algebraic and in polar forms. Find all complex numbers w such that $w^3 = z$, giving your answers in polar form.

Further questions

Q12 Prove that the addition and multiplication of complex numbers are

- (a) commutative and
- (b) associative.

Q13 Prove that the multiplication of complex numbers is distributive over addition, i.e.:

$$\forall v, w, z \in \mathbb{C} : v(w + z) = vw + vz \quad (\text{and } (v + w)z = vz + wz)$$

Q14 Using only the definition of multiplication of complex numbers prove that every nonzero complex number z has a multiplicative inverse, i.e. a complex number z^{-1} such that

$$zz^{-1} = z^{-1}z = 1.$$