Problem set 6: Functions

Summary of theory

function: A relation $f \subseteq X \times Y$ is a *function* if: $\forall x, y, y' : (x, y) \in f \land (x, y') \in f \Rightarrow y = y'$.
Notations:

- If f is a function then $f(x) = y \Leftrightarrow (x, y) \in f$.
- $X \to Y$ denotes the set of all functions $f \subseteq X \times Y$.
- $f \in X \to Y \Leftrightarrow f \subseteq X \times Y$ is a function.
- $f : X \to Y \Leftrightarrow (f \in X \to Y \land dmn(f) = X)$.

A function $f : X \to Y$ is

- *injective* if: $\forall x_1, x_2, y : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$;
- *surjective*, if $rng(f) = Y$ and
- *bijective*, if it is both injective and surjective.

Questions

Q1 In each of the following examples decide if the relation R is a function. If R is a function then determine the domain and range of R and decide whether R is surjective, injective and/or bijective.
(a) $A = \{1, 2, 3, 4, 5\}, B = \{10, 11, 12, 13, 14\}, R = \{(1, 11), (2, 11), (4, 12), (5, 10)\} \subseteq A \times B$
(b) $A = \{1, 2, 3, 4\}, B = \{a, b, c, d, e, f\}, R = \{(1, a), (2, c), (3, e), (3, f), (4, a)\} \subseteq A \times B$
(c) $A = \{1, 2, 3, 4, 5\}, B = \{a, b, c, d, e, f\}, R = \{(1, a), (4, e), (5, d)\} \subseteq A \times B$
(d) $A = \{1, 2, 3\}, B = \{1, 3, 5\}, R = \{(1, 1), (2, 5), (3, 5)\} \subseteq A \times B$

Q2
(a) Let $f : \mathbb{R} \to \mathbb{R}, f(x) := 3x - 4$. Prove that function f is bijective, and find the inverse of f.
(b) Let $g : \mathbb{R} \to \mathbb{R}, g(x) := 3 - |x|$. Prove that function g is neither injective, nor surjective.

Q3 In each of the following examples decide whether f is a function.
(a) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x \mid y$
(b) $f \subseteq \{0, 3, 5\} \times \{1, 2, 5\}, xfy \iff xy = 0$
(c) $f \subseteq \{1, 2, 5\} \times \{0, 3, 5\}, xfy \iff xy = 0$
(d) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff$ the set of digits contained by the base-10 form of x equals the set of digits in the base-10 form of y.
(e) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff 2x = y$
(f) $f \subseteq \mathbb{Z} \times \mathbb{Z}, xfy \iff x^2 = y^2$
(g) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x^2 = y^2$
(h) $f \subseteq \mathbb{R} \times \mathbb{R}, xfy \iff x^2 + y^2 = 9$

Q4 In each of the following examples decide if the given binary relation is a function.

(a) $f_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 7x = y^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(b) $f_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2 + 6y\} \subseteq \mathbb{R} \times \mathbb{R}$
(c) $f_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 7x^2 - 6 = y\} \subseteq \mathbb{R} \times \mathbb{R}$
(d) $f_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid y = |x|\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(e) $f_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = (x + 4)^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(f) $f_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid 2y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(g) $f_7 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 7 \mid x - y\} \subseteq \mathbb{Z} \times \mathbb{Z}$
(h) $f_8 = \{(x, y) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \mid xy = 1\} \subseteq (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$
(i) $f_9 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(j) $f_{10} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 3\} \subseteq \mathbb{Z} \times \mathbb{Z}$
(k) $f_{11} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y(1 - x^2) = x - 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(l) $f_{12} = \{(x, y) \in (\mathbb{R} \setminus \{1, -1\}) \times (\mathbb{R} \setminus \{1, -1\}) \mid y(1 - x^2) = x - 1\} \subseteq (\mathbb{R} \setminus \{1, -1\}) \times (\mathbb{R} \setminus \{1, -1\})$

About each relation that is a function decide if it is injective, surjective and/or bijective.

Further questions

Q5 Let $m \in \mathbb{R}^+$ and $A = \{$all isosceles triangles with height m (from base)$\}$, $B = \mathbb{R}^+$. Define the binary relation $R \subseteq A \times B$ as follows: $aRb, a \in A, b \in B$, if the area of a equals b. Show that R is a function, and examine the properties of f (i.e. decide if f is surjective, injective and/or bijective). **(1 point)**

Q6 Prove that the inverse of a function f is also a function if and only if f is injective.
. **(1 point)**