Problem set 5: Partial orders

Summary of theory

partial order, partially ordered set: A binary relation $\preceq\, \subseteq X \times X$ on set X is a *partial order* if it is reflexive, transitive and anti-symmetric. Then $(X, \preceq)$ is a *partially ordered set.*

Let $(X, \preceq)$ be a partially ordered set. An element $x \in X$ is called

- *greatest element* if $\forall y \in X : y \preceq x$.
- *maximal element* if $\nexists y \in X$: $y \neq x \wedge x \preceq y$.
- *least element* if $\forall y \in X : x \preceq y$.
- *minimal element* if $\nexists y \in X$: $y \neq x \wedge y \preceq x$.

immediate predecessor: Let $(X, \preceq)$ be a partially ordered set. For any $x, y \in X$, $x \neq y$ we say that x is an *immediate predecessor of* y if $\nexists z \in X : (z \neq x \wedge z \neq y \wedge x \preceq z \preceq y)$.

Hasse-diagram: A (finite) partially ordered set $(X, \preceq)$ can be represented on a Hasse-diagram as follows: each element of X is represented by a 'dot' on the diagram. Two dots representing x and y, respectively are connected by a line if and only if x is an immediate predecessor of y or y is an immediate predecessor of x. In this case we place the dot representing the element which is a predecessor of the other one lower on the diagram than the dot representing the other element.

comparable and incomparable elements: Two elements x and y of a partially ordered set $(X, \preceq)$ are said to be *comparable* if $x \preceq y$ or $y \preceq x$ holds; otherwise x and y are said to be *incomparable.*

(total) order: A partial order $\preceq\, \subseteq X \times X$ is called a *total order* (or *order* for short) if every pair of elements $x, y \in X$ is comparable, that is, if: $\forall x, y \in X : x \preceq y$ or $y \preceq x$. In other words: a *(total) order* is a dichotomous partial order. In this case the pair $(X, \preceq)$ is called a *totally ordered set* (or *ordered set* for short).

Questions

Q1 Let $A = \{2, 3, 6, 8, 9, 12, 18\} \subseteq \mathbb{N}^+$, $R \subseteq A \times A$ and $aRb \iff a \mid b$.
(a) Prove that R is a partial order on set A.
(b) Draw the Hasse-diagram of the partial order R.

Q2
(a) Prove that the relation $\preceq$ is a partial order on $\mathbb{N}$, where $\preceq$ is defined as follows: $\forall n, m \in \mathbb{N} : n \preceq m \iff \exists k \in \mathbb{N}$ such that $n + k = m$.
(b) Define the binary relation R on $\mathbb{N} \times \mathbb{N}$ as follows: $\forall m_1, m_2, n_1, n_2 \in \mathbb{N} : (m_1, n_1)R(m_2, n_2) \iff m_1 \leq m_2 \wedge n_1 \leq n_2$. Prove that R is a partial order on $\mathbb{N} \times \mathbb{N}$.

Q3 In each of the following examples decide if relation R is a partial order on the underlying set.
(a) P is the set of all polynomials with real coefficients and $R \subseteq P \times P$, $fRg \iff \deg f \leq \deg g$
(b) $R \subseteq \mathbb{Z} \times \mathbb{Z}, aRb \iff |a| \leq |b|$

(c) V is the set of all those vectors in $\mathbb{R}^2$ which are 10 units in length and $R \subseteq V \times V, xRy \iff$ the angle from the positive real axis to vector x is less than or equal to the angle from the positive real axis to vector y (we assume both of these angles to be in the interval $[0; 2\pi[$)

(d) $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, $xRy \iff$ the length of vector x is less than or equal to the length of vector y.

Q4 Decide which of the following relations are total orders on the set $A = \{1, 2, 3, 4\}$.
(a) $f = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
(b) $f = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
(c) $f = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,4)\}$

Further questions

Q5 Prove that the inverse of a partial order is also a partial order. **(1 point)**

Q6 Prove the following statements:
(a) The greatest (least) element in a partially ordered set (if it exists) is always a maximal (minimal) element. However, a maximal (minimal) element is not necessarily a greatest (least) element. **(1 point)**
(b) Every finite partially ordered set contains at least one maximal (minimal) element.
. **(1 point)**
(c) If a partially ordered set contains a greatest (or a least) element then it is unique. **(1 point)**
(d) In a totally ordered set an element is a maximal (minimal) element if and only if it is the greatest (least) element. **(1 point)**

Q7 Give an example for a partially ordered set which
(a) does not contain any maximal nor any minimal elements; **(1 point)**
(b) contains maximal (minimal) element(s), but no greatest (least) element. **(1 point)**
(c) contains more than one maximal (minimal) element. **(1 point)**