Problem set 4: Properties of homogeneous binary relations, equivalence relations and partitions

Summary of theory

Properties of homogeneous binary relations

homogeneous binary relation: A binary relation $R \subseteq X \times X$ is called a *homogeneous binary relation (on set X)* or a binary relation *on set X*.

A binary relation $R \subseteq X \times X$ on set X is:

- *transitive* if $\forall x, y, z \in X : (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$;
- *reflexive* if $\forall x \in X : (x, x) \in R$;
- *irreflexive* if $\forall x \in X : (x, x) \notin R$;
- *symmetric* if $\forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \in R$;
- *anti-symmetric* if $\forall x, y \in X : (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$;
- *strictly anti-symmetric* if $\forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \notin R$;
- *dichotomous* if $\forall x, y \in X$: at least one of $(x, y) \in R$ and $(y, x) \in R$ holds;
- *trichotomous* if $\forall x, y \in X$: exactly one of $(x, y) \in R$, $(y, x) \in R$ and $x = y$ holds.

Equivalence relations, equivalence classes, partitions

equivalence relation: A relation $\sim \subseteq X \times X$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

equivalence class: Let $\sim \subseteq X \times X$ be an equivalence relation on set X. The *equivalence class* of any $x \in X$ is: $\overline{x} = [x] = \{y \in X \mid y \sim x\}$ (i.e. the set of those elements in X which are $\sim$-related to x).

partition: Let $X \neq \emptyset$ be a set. A system of sets $\mathcal{P}$ is called a *partition* of X if: (1) all elements of $\mathcal{P}$ are nonempty subsets of X, (2) $\mathcal{P}$ is a pairwise disjoint system and (3) $\bigcup \mathcal{P} = X$. The elements of $\mathcal{P}$ are called the *blocks* or *cells* of the partition.

partition determined by an equivalence relation: Let $\sim \subseteq X \times X$ be an equivalence relation on a set $X \neq \emptyset$. The *partition determined by* $\sim$ (or the *quotient set of X by* $\sim$) is defined as: $X/\sim := \{[x] \mid x \in X\}$
(i.e. as the set of all equivalence classes of $\sim$).

equivalence relation determined by a partition: Let $\mathcal{P}$ be a partition of a set $X \neq \emptyset$. The *equivalence relation determined by* $\mathcal{P}$ is defined as: $\sim := \{(x, y) \mid x$ and y are contained by the same block of $\mathcal{P}\}$.

Questions

Properties of homogeneous binary relations

Q1 Let $X = \{1, 2, 3\}$. In each of the following examples below decide if the relation ρ on X is reflexive, symmetric, anti-symmetric and/or transitive.

(a) $\rho = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
(b) $\rho = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$
(c) $\rho = \{(1, 2), (1, 3), (2, 1), (3, 1)\}$
(d) $\rho = \{(1, 2), (2, 3), (3, 1)\}$

(e) $\rho = \{(1,2)\}$
(f) $\rho = \{(1,2),(2,1),(2,3),(3,2)\}$
(g) $\rho = \{(1,1),(2,2),(2,3),(3,3)\}$
(h) $\rho = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

Q2
(a) Can a relation be both symmetric and anti-symmetric at the same time? Can a relation be both reflexive and irreflexive? Justify your answers.
(b) Prove that if a relation is both symmetric and anti-symmetric then it is also transitive.
(c) Prove that if a relation that is not the empty set is both irreflexive and symmetric, then it is not transitive.

Q3 In each of the following examples decide if the given relation is reflexive, irreflexive, symmetric, anti-symmetric and/or transitive, and find the domain and the range of the relation.
(a) $R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \cdot b \text{ is odd}\}$ on $\mathbb{N}$.
(b) $S = \{(a,b) \in B \times B \mid$ the surname of a is shorter than the surname of $b\}$ on B where B is the set of all Discrete maths I students at ELTE.
(c) $T_X = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B \neq \emptyset\}$ on $\mathcal{P}(X)$ where X is a given set.
(d) $U = \{(x,y) \in C \times C \mid x$ touches y from inside$\}$ on C where C is the set of all circles in a given plane.

Q4 In each question below give an example for a relation on the set $\{1,2,3,4\}$ satisfying simultaneously all the properties listed:

(a) reflexive and not irreflexive
(b) anti-symmetric and not symmetric
(c) symmetric and not anti-symmetric
(d) both symmetric and anti-symmetric
(e) neither symmetric nor anti-symmetric
(f) both reflexive and trichotomous
(g) not reflexive, not transitive, not symmetric, not anti-symmetric and not trichotomous.

Equivalence relations and partitions

Q5 For part (a) and part (b) below, solve questions (1), (2) and (3).
(a) $\rho = \{(1,1),(1,5),(2,2),(3,3),(3,4),(4,3),(4,4),(5,1),(5,5)\}$ on set $A = \{1,2,3,4,5\}$.
(b) $\rho = \{(1,1),(1,5),(1,6),(1,8),(2,2),(2,4),(3,3),(3,7),(4,2),(4,4),(5,1),(5,5),(5,6),$
$(5,8),(6,1),(6,5),(6,6),(6,8),(7,3),(7,7),(8,1),(8,5),(8,6),(8,8)\}$ on set $A = \{1,2,3,4,5,6,7,8\}$.

(1) Prove that ρ is an equivalence relation on A.
(2) Draw the directed graph representation of ρ.
(3) Write down the partition of A determined by the equivalence relation ρ (in other words: Find the quotient set A/ρ).

Q6 For each partition of $\{a,b,c,d,e,f\}$ below, find the equivalence relation determined by the partition:

(a) $\{\{a,b,f\},\{c\},\{d,e\}\}$
(b) $\{\{a\},\{b\},\{c\},\{d\},\{e,f\}\}$

Q7 In each of the following examples prove that R is an equivalence relation (on the set which R is defined on in the example), and find the equivalence classes of R.
(a) $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m+n \text{ is even}\}$
(b) $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m-n \text{ is divisible by } 3\}$
(c) $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2+y^2 \text{ is even}\}$
(d) * $R = \{(a,b) \in \mathbb{R} \times \mathbb{R} \mid a-b \text{ is rational}\}$
(e) $R = \{((x_1,y_1),(x_2,y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1+y_1 = x_2+y_2\}$
(f) $R = \{((x_1,y_1),(x_2,y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \cdot y_1 = x_2 \cdot y_2\}$

Further questions

Q8 Let $R, S \subseteq A \times A$ be symmetric relations. Prove that $R \circ S$ is symmetric if and only if $R \circ S = S \circ R$. **(1 point)**

Q9 Let X be a set.
(a) Prove that the intersection of two reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric, transitive) relations on X is also a(n) reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric, transitive) relation. (each property: **0.5 point)**
(b) Is the statement also true for the union (instead of the intersection) of the two relations? Prove your answers. (each property: **0.5 point)**

Q10 How many different

(a)	reflexive	(c)	symmetric	(e)	strictly anti-symmetric	(g)	trichotomous
(b)	irreflexive	(d)	anti-symmetric	(f)	dichotomous		

binary relations exist on an n-element set? Prove your answers. (Note: There is no general formula known for the number of transitive relations on an n-element set!) (each property: **1 point)**

Q11 Write a program the input of which is a finite set A and a binary relation ρ on A, and the program decides about ρ if it is

(a)	reflexive	(c)	symmetric	(e)	strictly anti-symmetric	(g)	trichotomous
(b)	irreflexive	(d)	anti-symmetric	(f)	dichotomous	(h)	transitive

(each property: **1 point)**

Q12 Write a program whose inputs are a finite set A and a binary relation ρ on A, and the program decides about ρ if it is an equivalence relation on A; if ρ is an equivalence relation then the program finds the partition A/ρ of A determined by ρ. **(2 points)**