

Problem set 2: Sets

Summary of theory

Some special sets

empty set: The *empty set* (i.e. the set that has no elements) is denoted by $\emptyset = \{\}$ or $\emptyset$.

system of sets: A set such that all of its elements are also sets is called a *system of sets*.

Subsets, proper subsets

subset: A set A is a *subset* of set B , in notation: $A \subseteq B$, if all elements of A are also elements of set B , that is, if: $\forall x(x \in A \Rightarrow x \in B)$.

proper subset: If A is a subset of B and $A \neq B$, then A is a *proper subset* of B , in notation: $A \subsetneq B$.

Set union and set intersection

set union: The *union* $A \cup B$ of sets A and B is the set which contains all elements of A and B , that is: $A \cup B = \{x \mid x \in A \vee x \in B\}$. In general: Let $\mathcal{A}$ be a system of sets. Then $\bigcup \mathcal{A}$ is the set which contains all elements of the elements of $\mathcal{A}$, that is: $\bigcup \mathcal{A} = \{x \mid \exists A \in \mathcal{A} : x \in A\}$.

set intersection: The *intersection* $A \cap B$ of sets A and B is the set containing exactly the common elements of A and B , that is: $A \cap B = \{x \mid x \in A \wedge x \in B\}$. In general: Let $\mathcal{A}$ be a system of sets. Then the intersection $\bigcap \mathcal{A}$ is the set which contains those elements which are elements of all sets in $\mathcal{A}$, that is: $\bigcap \mathcal{A} = \{x \mid \forall A \in \mathcal{A} : x \in A\}$.

Properties of set union and set intersection: For every set A, B and C :

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| 1. $A \cup \emptyset = A$ | 6. $A \cap \emptyset = \emptyset$ |
| 2. $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity) | 7. $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity) |
| 3. $A \cup B = B \cup A$ (commutativity) | 8. $A \cap B = B \cap A$ (commutativity) |
| 4. $A \cup A = A$ (idempotence) | 9. $A \cap A = A$ (idempotence) |
| 5. $A \subseteq B \Leftrightarrow A \cup B = B$ | 10. $A \subseteq B \Leftrightarrow A \cap B = A$ |

Distributive properties of set union and set intersection: For every set A, B and C :

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|---|---|
| 1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | 2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
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Set difference and set complement

set difference: The *difference* of sets A and B is the set $A \setminus B = \{x \in A : x \notin B\}$.

set complement: Let U be the universal set. Then for every set $A \subseteq U$ the *complement* of A is defined as $\bar{A} = A' = U \setminus A$.

Expressing set difference using set intersection and complement: Suppose a universal set is given. Then for every set A and B : $A \setminus B = A \cap \bar{B}$.

Properties of set complement: Let U be the universal set. Then for every set A and B :

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|----------------------------|-----------------------------------|--|---|
| 1. $\bar{\bar{A}} = A$; | 3. $\bar{U} = \emptyset$; | 5. $A \cup \bar{A} = U$; | 7. $\overline{A \cap B} = \bar{A} \cup \bar{B}$; |
| 2. $\bar{\emptyset} = U$; | 4. $A \cap \bar{A} = \emptyset$; | 6. $A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$; | 8. $\overline{A \cup B} = \bar{A} \cap \bar{B}$. |

Properties 7 and 8 are called *De Morgan's laws (for sets)*.

Symmetric difference of sets

symmetric difference: The *symmetric difference* of sets A and B is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Alternative expression of the symmetric difference: For every set A and B : $A \triangle B = (A \cup B) \setminus (B \cap A)$.

Cartesian product of sets

Cartesian product: The *Cartesian product* of sets A and B is: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Power set of a set

power set: The *power set* $\mathcal{P}(A)$ of a set A is the set of all subsets of A , that is $\mathcal{P}(A) = \{S \mid S \subseteq A\}$.

Questions**Elements and subsets of sets and set operations $\cup$, $\cap$, $\setminus$ and set complement**

Q1 Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set, $A = \{x \in \mathbb{N} \mid 1 \leq x \leq 4\}$, $B = \{0, 2, 4, 8\}$ and $C = \{2, 3, 5, 7\}$.

(a) Write down the following using roster notation, i.e. by listing all their elements between curly braces $\{\}$:

$$(i) \ A \cap B \qquad (ii) \ B \cup C \qquad (iii) \ A \setminus C \qquad (iv) \ \overline{C}$$

(b) Consider the systems of sets $X = \{A, B, C\}$ and $Y = \{\{0, 2, 4, 6, 8\}, \{1, 3, 5, 7, 9\}\}$. Find the following sets:

$$(i) \ \cap X \qquad (ii) \ \cup X \qquad (iii) \ X \cup Y \qquad (iv) \ X \cap Y$$

(c) Determine the truth value of each of the following statements:

$$\begin{array}{llll} (i) \ 4 \in B & (iv) \ 3 \in A \cap B & (vii) \ A \subseteq \cup Y & (x) \ \{2\} \subseteq A \\ (ii) \ A \subseteq B & (v) \ \{1, 2\} \subseteq A & (viii) \ C \cap \emptyset = \emptyset & (xi) \ 2 \in \cup X \\ (iii) \ \{\emptyset\} \subseteq \cup X & (vi) \ A \in \cup Y & (ix) \ 2 \subseteq A & (xii) \ \{2\} \in \cap X \end{array}$$

Q2 Let $\mathcal{A} = \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$. Find the sets $\cup \mathcal{A}$ and $\cap \mathcal{A}$.

Q3 Consider the system of sets $X = \{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{0, 2, 3, 7\}\}$. Find the following sets:

$$\begin{array}{ll} (a) \ \cap X & (b) \ X \cup \{5, 6, 7, 8\} \\ (c) \ X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\} & (d) \ \cup (X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\}) \\ (e) \ \cap (X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\}) & \end{array}$$

Q4 Find all possible sets A , B and C that satisfy all the following properties:

$A \setminus B = \{1, 3, 5\}$, $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$, $(A \cap C) \cup (B \cap C) = \emptyset$, $C \setminus B = \{2, 4\}$ and $(A \cap B) \setminus C = \{6\}$.

Q5 Decide whether there exist sets A , B and C satisfying all the conditions below. If such sets exist, give an example, otherwise prove that such sets do not exist:

$$A \cap B \neq \emptyset, \quad A \cap C = \emptyset, \quad (A \cap B) \setminus C = \emptyset.$$

Proving or disproving set identities

Q6 Let $A = \{a, b, c, d\}$, $B = \{c, d\}$ and $C = \{a, c, e\}$. Show that then $A \setminus (B \setminus C) = (A \setminus B) \cup (B \cap C)$. Is this statement true for all sets A, B and C ?

Q7 Using only the definitions of set operations and the properties of logical operations learnt, prove that the following equalities are true for any universal set U and sets $A, B, C \subseteq U$. (Hence, these equalities are identities.)

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| (a) $A \cup B = B \cup A$ | (g) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ |
| (b) $(A \cup B) \cup C = A \cup (B \cup C)$ | (h) $\overline{A \cap B} = \overline{A} \cup \overline{B}$ |
| (c) $A \cap B = B \cap A$ | (i) $A \cup \overline{A} = U$ |
| (d) $(A \cap B) \cap C = A \cap (B \cap C)$ | (j) $A \cap \overline{A} = \emptyset$ |
| (e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | (k) $\overline{\overline{A}} = A$ |
| (f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | (l) $A \setminus B = A \cap \overline{B}$ |

In the following questions we may use the properties of set operations learnt in the lecture, including those proved in **Q7** above.

Q8 Prove that the following equalities hold for every set A and B :

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| (a) $(A \setminus B) \cap B = \emptyset$ | (b) $(A \cup \overline{B}) \cap (\overline{A} \cup \overline{B}) = \overline{B}$ |
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Q10 Write the following expression in its simplest possible form, proving the equality of the two expressions for all sets A, B and C :

$$(A \cup (A \cap B) \cup (A \cap B \cap C)) \cap (A \cup B \cup C).$$

Q11 Prove that the following equalities hold for all universal sets U and sets $A, B, C \subseteq U$. (Hence, these equalities are identities.)

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|---|--|
| (a) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$ | (c) $A \setminus (A \setminus (B \setminus C)) = A \cap B \cap \overline{C}$ |
| (b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ | |

Q12 Prove the following identity: $\overline{(\overline{A \cap B \cup C}) \cap \overline{A} \cup \overline{B} \cup \overline{C}} = A \cup \overline{B} \cup \overline{C}$.

Q13 Decide which of the following statements are true for all sets A, B and C . Prove your answers.

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|--|---|
| (a) $\overline{A} \cap B = B \setminus A$ | (d) $(A \cup B) \setminus A = B$ |
| (b) $(A \cap B) \setminus C = (A \setminus B) \cap C$ | (e) $(A \cup B) \setminus C = A \cup (B \setminus C)$ |
| (c) $(A \cup B) \cap (B \setminus A) = (A \cup B) \setminus (A \setminus B)$ | |

Symmetric difference of sets

Q14 Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:

- (a) $A \triangle B$ (b) $A \triangle C$ (c) $(A \triangle B) \triangle C$ (d) $A \triangle (B \triangle C)$

Q15 Prove the following identities.

- (a) $A \triangle \emptyset = A$ (c) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$
 (b) $A \triangle A = \emptyset$ (d) $A \triangle (A \triangle B) = B$

Cartesian product of sets

Q16 Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:

- (a) $A \times B$ (c) $A \times A$ (e) $A \times (A \times B)$
 (b) $B \times A$ (d) $(A \times A) \times B$ (f) $A \times A \times B$

Q17 Prove that for every nonempty set A, B, C and D , $A \times B \subseteq C \times D$ holds if and only if $A \subseteq C$ and $B \subseteq D$.

Q18 Prove that the following is true for all sets A, B and C :

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$

Power set of a set

Q19 Find the power set $\mathcal{P}(A)$ of A when

- (a) $A = \{a, b\}$ (b) $A = \{a, b, c\}$ (c) $A = \{1, 2, 3\}$ (d) $A = \{a, b, c, d\}$

Q20 Prove that for every sets A and B we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, where $\mathcal{P}(A)$ denotes the power set of A . What can we say about the truth value of the statement obtained by replacing $\cap$ by $\cup$?

Further questions

Q21 Based on **Q15** part c, for every set A, B and C we have $A \triangle (B \triangle C) = (A \triangle B) \triangle C$. Therefore the notation $A \triangle B \triangle C := A \triangle (B \triangle C) = (A \triangle B) \triangle C$ can be introduced. Prove that for every set A, B, C and D we have $A \triangle (B \triangle C \triangle D) = (A \triangle B \triangle C) \triangle D$. (1 point)

Q22 Natural numbers can be defined by sets recursively as follows:

- $0 := \emptyset$ and
- $\forall n \in \mathbb{N}, n > 0 : n + 1 := n \cup \{n\}$.

(For example, by the above definition: $1 := 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}$ and $2 = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.) Write down the numbers 3 and 4 represented as sets according to the above definition.

(1 point)