Problem set 1: Logic

Summary of theory

Some logical operators

$\neg$	negation
$\vee$	inclusive or (disjunction)
$\wedge$	and (conjunction)
$\oplus$	exclusive or
$\parallel$	conflicting or
$\Rightarrow$	'If ... then ...' (implication, conditional)
$\Leftrightarrow$	'If and only if' (equivalence, biconditional)

Logical operators are defined by their truth tables:

A	$\neg A$
T	F
F	T

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$A \parallel B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	T	T	F	F	T	T
T	F	F	T	T	T	F	F
F	T	F	T	T	T	T	F
F	F	F	F	F	T	T	T

Theorem 1 (Properties of logical operations/Laws of 0$^{\text{th}}$ order logic) *For every proposition A, B and C the following statements are always true (i.e. these statements are tautologies):*

Properties of $\vee$ and $\wedge$:

1. idempotence:	$A \vee A \Leftrightarrow A$	$A \wedge A \Leftrightarrow A$
2. associativity:	$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$	$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$
3. commutativity:	$(A \vee B) \Leftrightarrow (B \vee A)$	$(A \wedge B) \Leftrightarrow (B \wedge A)$
4. absorption laws:	$(A \vee B) \wedge A \Leftrightarrow A$	$(A \wedge B) \vee A \Leftrightarrow A$
5. De Morgan's laws:	$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$	$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$
6. distributivity:	$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

Inference rules:

7. law of contrapositive:	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
8. modus ponens:	$((A \Rightarrow B) \wedge A) \Rightarrow B$
9. syllogism:	$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
10.	$((A \Rightarrow B) \wedge (B \Rightarrow A)) \Leftrightarrow (A \Leftrightarrow B)$

Questions

Propositional calculus (0^{th} order logic): propositional formulas (0^{th} order formulas)

Q1 Consider the following four propositions:

A: The capital city of Egypt is Cairo.
B: $3 + 2 = 9$.
C: The largest ocean in the world is the Atlantic ocean.
D: Pigeons are birds.

About each of the following statements decide if it is true or false:

a) $A \vee B$ b) $A \wedge B$ c) $\neg D$ d) $B \oplus C$ e) $B||C$ f) $A \Rightarrow B$ g) $B \Rightarrow A$
h) $B \Rightarrow C$ i) $A \Leftrightarrow D$ j) $(B \vee C) \wedge (D \Rightarrow B)$ k) $(\neg A \wedge D) \Leftrightarrow \neg(B \wedge C)$

Q2 Prove the properties of logical operators formulated in Theorem 1 'Properties of logical operations/Laws of 0^{th} order logic'.

Predicate calculus (1^{st} order logic): predicate formulas (1^{st} order formulas)

Q3 On the positive integers denote by $P(x)$, $E(x)$, $O(x)$ and $D(x, y)$ the predicates that x is prime, x is even, x is odd, and that x is a divisor of y, respectively.

a) $P(7)$;
b) $E(2) \wedge P(2)$;
c) $\forall x(D(2, x) \Rightarrow E(x))$;
d) $\exists x(E(x) \wedge D(x, 6))$;
e) $\forall x(\neg E(x) \Rightarrow \neg D(2, x))$;
f) $\forall x(E(x) \Rightarrow (\forall y(D(x, y) \Rightarrow E(y))))$;
g) $\forall x(P(x) \Rightarrow (\exists y(E(y) \wedge D(x, y))))$;
h) $\forall x(O(x) \Rightarrow (\forall y(P(y) \Rightarrow \neg D(x, y))))$;
i) $(\exists x(E(x) \wedge P(x))) \wedge (\neg(\exists x(E(x) \wedge P(x) \wedge (\exists y(\neg x = y \wedge E(y) \wedge P(y))))))$.

For each of the above formulas:

1. Translate the formula into an everyday English sentence.
2. Decide if the statement is true.
3. Write down the negation of each statement using a formula and using an English sentence.

Q4 On the set of all people in the world denote by $Law(x)$, $J(x)$, $C(x)$, $O(x)$, $Liv(x)$, $P(x)$, $Rep(x)$, $W(x)$, $S(x, y)$, and $Res(x, y)$ respectively, the statement that x is a lawyer, a judge, crafty, old, lively, a politician, a representative, a woman, that x is a spouse of y, and that x respects y, respectively. Formalize the following statements:

a) every judge is a lawyer;
b) there are crafty lawyers;
c) there are no crafty judges;
d) some judges are old, but lively;
e) judge d is neither old nor lively;
f) every lawyer who has a lively wife, is a representative;
g) except for judges, all lawyers are crafty;
h) every representative who has a wife is lively;
i) no representative has an old wife;
j) some women are both lawyers and representatives;
k) every old representative is a lawyer;
l) some lawyers who are politicians, are representatives;
m) some lawyers respect only judges;
n) every woman who is a lawyer, respects some judges;
o) there is a judge who respects some women;
p) some crafty people do not respect any judges;
q) judge d does not respect any crafty person;
r) some judges and some crafty people respect judge d;
s) only judges respect judges;
t) judges respect only judges.

Q5 On the set of all people in the world denote by $W(x)$ and $C(x, y)$, respectively, that x is a woman and that x is a child of y. Formalize the following statements: x is the

a) daughter b) parent c) father d) mother e) grandchild f) grandparent
g) grandfather h) grandmother i) paternal grandfather j) maternal grandfather
k) paternal grandmother l) maternal grandmother m) sibling n) brother
o) sister p) cousin q) uncle r) aunt s) nephew t) niece

of y, respectively.

Further questions

Q6 There are 50 identical coins lying on a table: 25 with the head and 25 with the tail up, respectively. With your eyes blindfolded, how can you divide them into two groups such that both groups have the same number of coins heads up, if you are allowed to turn over any coins any number of times? The head and tail sides of each coin feel the same to the touch, hence you cannot tell the difference between the two sides blindfolded. **(1 point)**

Q7 (SAT-problem) Let $x_1, x_2, \ldots$ be Boolean variables, and consider the formulas that can be formed using some of these variables and the logical operations $\wedge$, $\vee$ and $\neg$. A formula is said to be *satisfiable* if the values of the variables $x_1, x_2, \ldots$ can be chosen so that the value of the formula becomes true. For example, $F(x_1, x_2) = x_1 \wedge x_2$ is satisfiable, since for $x_1 = x_2 = true$, F is *true*. The formula $G(x_1) = x_1 \wedge \neg x_1$ is not satisfiable, since $G(true) = G(false) = false$. Write a program, which can decide about any given n-variable formula, whether it is satisfiable. Which is the largest n (where n is the number of Boolean variables in the formula) such that your program can decide about every n-variable formula if it is satisfiable, within a reasonable time limit (e.g. in 1 hour)? **(2 points)**