Algebraic structures

(exercises)

Application of discrete models

1. Write a function for each of the following exercises. Each function gets a list, whose elements form a ring. Don't assume anything about the elements other than they have the two ring operations (`a + b` and `a * b`) and that they can be compared for equality (`a == b`, `a != b`).
   a) Decide whether the ring is commutative.
   Try it for $\mathbb{Z}_{10}$ and $\mathbb{Z}_3^{2\times 2}$.
   b) Return the zero element of the ring.
   Try it for $\mathbb{Z}_{10}$ and $\mathbb{Z}_3^{2\times 2}$.
   c) Return the identity element of the ring if it has one, otherwise return `None`.
   Try it for $\mathbb{Z}_{10}$, $2\mathbb{Z}_{20}$ (i.e. the even elements of $\mathbb{Z}_{20}$) and $2\mathbb{Z}_{10}$.
   d) Return a zero divisor in the ring if it has one, otherwise return `None`.
   Try it for $\mathbb{Z}_5$ and $\mathbb{Z}_6$.
   e) Decide whether the ring is an integral domain.
   Which of $\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_{20}$ are integral domains?

2. Is there any quadratic polynomial in $\mathbb{Z}_6[x]$ for which all six elements of $\mathbb{Z}_6$ are roots?

3. For all finite fields with ≤ 10 elements, calculate the polynomial for which all elements of the field are roots (as simple roots, and without any extra factors).

4. Implement the field operations of the field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ as follows. Each $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ element is represented by the pair (a, b).
   a) Write a function that calculates the sum of two elements elements of $\mathbb{Q}(\sqrt{2})$.
   (For example: `add((2, 1/2), (1/3, 1)) == (7/3, 3/2)`.)
   b) Write a function that calculates the product of two elements elements of $\mathbb{Q}(\sqrt{2})$.
   c) Write a function that calculates the multiplicative inverse of a nonzero element of $\mathbb{Q}(\sqrt{2})$.

   Write these functions in such a way that they work for not only $\mathbb{Q}$, but for any other field which contains no $\sqrt{2}$, e.g. $\mathbb{Z}_3$. Using this, calculate the multiplicative inverse of $\overline{1} + \overline{2}\sqrt{2} \in \mathbb{Z}_3(\sqrt{2})$, and check the result using multiplication.