SageMath introduction

(exercises)

Application of discrete models

1. Calculate the exact value:
   a) 2^{100}; b) $2^{2^{10}}$; c) $\left(2^2\right)^{10}$; d) $2^{\left(2^{10}\right)}$; e) $2^{2^{2^2}}$; f) $2^{2^{2^{2^2}}}$; g) $2^{\left(2^{2^2}\right)^2}$;
   h) $1024!$; i) $(5!)!$; j) $((3!)!)!$; k) $((((2!)!)!)!)!$; l) $\binom{100}{10}$; m) $\binom{\binom{10}{3}}{\binom{10}{2}}$.

2. Calculate the exact value:
$$\sqrt{\frac{\left(\operatorname{tg}\left(-\frac{7}{12}\pi\right)-2\right)^2}{\frac{123456}{\left(\frac{1}{2}\right)^{-13}+9!-8\cdot 88}-2^{\binom{42}{5}-(2^8+1)\cdot 3310}}}$$

3. Calculate numerically:
   a) $3+\dfrac{1}{7}$; b) $3+\dfrac{1}{7+\frac{1}{15}}$; c) $3+\dfrac{1}{7+\frac{1}{15+\frac{1}{1}}}$; d) $3+\dfrac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292}}}}$.

4. Calculate:
   a) $\sqrt{-64}$; b) $\dfrac{4+3i}{(2-i)^2}$; c) $\dfrac{(1+\sqrt{3}i)^{18}}{(\sqrt{2}-\sqrt{2}i)^{15}}$; d) $\dfrac{\overline{1-i}}{(2+i)^2}-\dfrac{1}{3-i}$.

5. Simplify:
$$\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}+\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\cdot\frac{1-x^2}{x^2+2x+1}$$

6. Solve (and check the result):
   a) $x^2-3x+2=0$; b) $x^2-x-1=0$; c) $x^2-6x+13=0$; d) $x^2+x+1=0$;
   e) $x^3-10x^2+34x=0$; f) $x^{12}=1$; g) $x^3=-64$;
   h) $x^7+3x^6-10x^5-18x^4+29x^3-9x^2+40x+12=0$;
   i) $x^7+3x^6-10x^5-18x^4+28x^3-9x^2+40x+12=0$.

7. Solve parametrically for x:
   a) $\dfrac{x}{a}+\dfrac{x}{b}=1$; b) $ax+b=c$; c) $ax^2+(2-a^2)x-2a=0$.

8. Calculate the general formula for
   a) quadratic equations; b) cubic equations; c) quartic equations.

9. Solve:
   a) $\begin{cases} 2x+3y-z &= 5\\ 3x-2y+z &= 2\\ x+y-z &= 0; \end{cases}$ b) $\begin{cases} 4x+y+5z &= 5\\ 3x+2y+2z+w &= 4\\ 2x+3y-z+2w &= 3. \end{cases}$

10. Calculate the general formula for systems of linear equations with two variables.

11. Bring to a closed form (i.e. without $\sum$ or $\prod$):

    a) $\sum_{k=1}^{n} k$; b) $\sum_{k=1}^{n} k^2$; c) $\sum_{k=1}^{n+1} \binom{k}{2}$; d) $\sum_{k=1}^{n} \sum_{j=1}^{k} j$; e) $\prod_{k=1}^{n} k^2$;

12. Prove:
$$\sum_{k=1}^{n} \frac{1}{(4k-3)(4k+1)} = \frac{n}{4n+1}.$$

13. Calculate:

    a) $\lim_{x\to\infty} \frac{x^4 - x^3 - 3}{2x^4 + 3x^2 + 1}$; b) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$; c) $\left(\frac{x+2}{x^2+3}\right)'$; d) $\left(\left(1 + \frac{1}{x}\right)^x\right)'$;

    e) $\sin(2x)'$; f) $\sin(2x)''$; g) $\sin(2x)'''$; h) $\int_0^1 x^2 dx$; i) $\int \ln x \, dx$; j) $\int_5^7 \frac{x+2}{x^2+3} dx$.

14. Let

    - $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, • $B = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 7 \end{bmatrix}$, • $C = \begin{bmatrix} 1 & 3 & 1 \\ 3 & -1 & 2 \end{bmatrix}$.

    Calculate the inverse of $(A - B) \cdot C^T$.

15. Calculate the determinant of 2×2 and 3×3 matrices in general.

16. Calculate the sum of integers from 1 to 100 – in at least three ways.

17. Create a list with the following elements:
    a) the first 20 factorials: $[1, 2, 6, 24, ...]$;
    b) the 10^{th} line of Pascal's triangle;
    c) 10 lists, where the n^{th} list is the n^{th} line of Pascal's triangle: $[[1], [1, 1], [1, 2, 1], \ldots]$;
    d) the sines of the degrees $0°, 30°, \ldots, 360°$;
    e) the powers of x's up to 100 $[1, x, x^2, x^3, \ldots, x^{100}]$;
    f) $\sin(2x)$ and its first 20 derivaties (can you guess the rule?).

18. Create the following matrices:
    a) times table (i.e. $a_{ij} = i \cdot j$) from 1 to 10;
    b) first 16 lines of Pascal's triangle (extended by 0's to a square).

19. a) Write a function that, for a given n, calculates the polynomial $1 + x + x^2 + x^3 + \ldots + x^{n-1}$.
    b) Call it for $n = 100$, and multiply it by $(x - 1)$ (and simplify it). What do we get? Why?
    c) Print the factorization of the polynomials from a) from $n = 1$ to 20. What is the connection between the reducibility of the n^{th} polynomial and the properties of n?.
    d) Print the results of the built-in function `cyclotomic_polynomial(n)` from $n = 1$ to 20. What does it have to do with the above?
    e) What is the n^{th} derivative of the $(n + 1)^{\text{st}}$ polynomial? What is the rule? Why?

20. a) Write a function that prints the n^{th} complex roots of unity for a given n.
    b) Print the results from $n = 1$ to 12.

c) What does it have to do with the previous exercise?

d) Write another function that prints only the primitive roots of unity. (Whose roots are they?)

21. Find an integer b for which the equation $x^3 + bx^2 + 12x - 8 = 0$ has exactly one solution.

22. Find a monic quadratic equation with integer coefficients (i.e. $x^2 + bx + c = 0$, where $b, c \in \mathbb{Z}$) which is satisfied by the golden ratio $\left(\phi = \dfrac{\sqrt{5}+1}{2}\right)$.

23. Calculate numerically the first 20 elements of the following sequence:

   • $1+1$; • $1+\dfrac{1}{1+1}$; • $1+\dfrac{1}{1+\frac{1}{1+1}}$; • $1+\dfrac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$; • ....

   Can you guess their limit?

24. Calculate numerically the following partial sums from $n = 1$ to 20.
$$\sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$
   Can you guess their limit? Calculate the sum symbolically to ∞.

25. Write a function that calculates the Vandermonde matrix for a given input vector, e.g. for a vector (a_1, a_2, a_3, a_4), it gives the following matrix:
$$\begin{pmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{pmatrix}.$$
   Call this function with the vector (a, b, c, d, e) of symbolic variables. Calculate the determinant of the matrix, and factor the result. Try this with various vector sizes. What is the rule?