Set theory

Discrete mathematics I – exercises

1. Write the following sets formally, using set-builder notation:
   a) set of even numbers; b) set of positive numbers not divisible by 3;
   c) set of square numbers; d) set of rational numbers; e) set of prime numbers;
   f) set of pairs of adjacent natural numbers.

2. Let:
   - $A = \{x \in \mathbb{Z} \mid 1 \leq x \leq 4\}$; • $B = \{0, 2, 4, 8\}$; • $C = \{\text{one-digit prime numbers}\}$;
   - $X = \{A, B, C\}$.

   Calculate / decide:
   a) $A \cap B$; b) $B \cup C$; c) $A \setminus C$; d) $1 \in C$; e) $3 \in A \cap B$; f) $A \subseteq B$;
   g) $\cap X$; h) $\cup X$; i) $\{2\} \subseteq A$; j) $2 \subseteq A$; k) $A \in X$; l) $A \subseteq X$; m) $2 \in \emptyset$;
   n) $A \subseteq \emptyset$; o) $\emptyset \subseteq A$; p) $A \subseteq A$; q) $A \in A$; r) $A \in \{\{A\}, B\}$; s) $A \subseteq \{\{A\}, B\}$;
   t) $\emptyset = \{\emptyset\}$; u) $B \cup \emptyset$; v) $B \cup \{\emptyset\}$.

3. Let $X = \{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{0, 2, 3, 7\}\}$. Calculate:
   a) $\cap X$; b) $X \cup \{1, 2, 3\}$; c) $X \cup \{\{3, 5, 7\}, \{1, 2, 3\}\}$;
   d) $\cup (X \cup \{\{3, 5, 7\}, \{1, 2, 3\}\})$; e) $\cap (X \cup \{\{3, 5, 7\}, \{1, 2, 3\}\})$.

4. Are there sets A, B and C for which all statements are true:
   - $A \cap B \neq \emptyset$, • $A \cap C = \emptyset$, • $(A \cap B) \setminus C = \emptyset$?

5. Find A, B and C for which all statements are true:
   - $A \setminus B = \{1, 3, 5\}$, • $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$, • $(A \cap C) \cup (B \cap C) = \emptyset$,
   - $(A \cap B) \setminus C = \{6\}$, • $C \setminus B = \{2, 4\}$.

6. Prove for any A, B, C sets:
   a) $A \cup B = B \cup A$; b) $(A \cup B) \cup C = A \cup (B \cup C)$;
   c) $A \cap B = B \cap A$; d) $(A \cap B) \cap C = A \cap (B \cap C)$;
   e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
   f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
   g) $\overline{A \cup B} = \overline{A} \cap \overline{B}$; h) $\overline{A \cap B} = \overline{A} \cup \overline{B}$; i) $A \cup \overline{A} = U$ (where U is the universal set);
   j) $A \cap \overline{A} = \emptyset$; k) $\overline{\overline{A}} = A$.

7. Prove that:
   a) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$; b) $A \setminus (B \cup C) = (A \setminus B) \setminus C$.

8. Which are true for all sets A, B and C:
a) $(A \cup B) \cap A = (A \cap B) \cup A$; b) $A \setminus (A \cap B) = A \setminus B$; c) $(A \cup B) \setminus A = B$;
d) $(A \cup B) \setminus C = A \cup (B \setminus C)$; e) $(A \setminus B) \cap C = (A \cap C) \setminus B = (A \cap C) \setminus (B \cap C)$?

9. Prove that:
a) $A \cap B \subseteq C \iff A \subseteq \overline{B} \cup C$; b) $(A \cap B) \cup C = A \cap (B \cup C) \iff C \subseteq A$;
c) $(A \setminus B) \cup B = A \iff B \subseteq A$; d) $A \cup B = A \iff A \cap B = B$;
e) $A = \emptyset \iff B = A \triangle B$; f) $A \triangle B = C \iff B \triangle C = A$.

10. Let:
- $A = \{1, 2\}$; • $B = \{a, b, c\}$; • $C = \{2, 3, 4\}$.

Calculate:
a) $A \times A$; b) $A \times B$; c) $B \times A$; d) $A \times A \times B$; e) $(A \times A) \times B$; f) $A \times (A \times B)$.

11. Let:
- $A = \{1, 2\}$; • $B = \{2, 3, 4\}$; • $C = \{1, 3, 5\}$.

Calculate:
a) $A \triangle B$; b) $A \triangle C$; c) $(A \triangle B) \triangle C$; d) $A \triangle (B \triangle C)$.

12. Prove that:
a) $A \triangle \emptyset = A$; b) $A \triangle A = \emptyset$; c) $A \triangle (A \triangle B) = B$; d) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

13. From the previous exercise, we have $A \triangle (B \triangle C) = (A \triangle B) \triangle C$, i.e. the symmetric difference is associative. Therefore, we can omit the parentheses and simply write $A \triangle B \triangle C$. Prove that $A \triangle (B \triangle C \triangle D) = (A \triangle B \triangle C) \triangle D$.

14. Calculate:
a) $2^{\emptyset}$; b) $2^{\{a\}}$; c) $2^{\{a,b\}}$; d) $2^{\{a,b,c\}}$; e) $2^{2^{2^{\emptyset}}}$.

15. Natural numbers can be built using only set theory by the following recursion:
- $0 := \emptyset$,
- $\forall n \in \mathbb{N} : n + 1 := n \cup \{n\}$.

Write down 1, 2, 3 and 4 in their set representation.