Relations

Discrete mathematics I – exercises

1. Let
   - $A = \{1, 2, 3, 4\}$, • $B = \{5, 6, 7, 8, 9\}$,
   - $R \subseteq A \times B,\ R = \{(1, 5), (1, 6), (1, 7), (3, 6), (3, 9), (4, 5), (4, 7), (4, 9)\}$.

   a) Draw R on an arrow diagram. b) What is the domain and range of R?
   c) Restrict R to the set $\{1, 2, 3\}$. d) Restrict R to the set $\{4\}$.
   e) Find R^{-1}. f) Find the image $R(\{1, 2\})$. g) Find the inverse image $R^{-1}(\{5, 6\})$.

2. Let A, B, R as above. Which of the following relations are extensions of R:
   a) $\{(1, 5), (1, 6), (1, 7), (2, 2), (2, 4), (3, 6), (3, 9), (4, 3), (4, 5), (4, 7), (4, 9)\}$;
   b) $\{(1, 5), (1, 6), (1, 7), (3, 6), (3, 8), (4, 5), (4, 6), (4, 7), (4, 9)\}$; c) $A \times B$; d) $B \times A$?

3. Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ and $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 2b\}$.
   a) What is the domain and range of R? b) Find R^{-1}.
   c) $R(\{3, 4, \dots, 10\}) = ?$ d) Restrict R to $\{1, 2, \dots, 6\}$.

4. What is $S \circ R$ if:
   - $A = \{1, 2, 3\}, B = \{a, b, c, d, e, f\}, C = \{2, 4, 6, 8\}$;
   - $R \subseteq A \times B,\ R = \{(1, a), (1, b), (2, c), (2, f), (3, d), (3, e), (3, f)\}$;
   - $S \subseteq B \times C,\ S = \{(a, 2), (a, 4), (c, 6), (c, 8), (d, 2), (d, 4), (d, 6), (f, 8)\}$.

5. Calculate $S \circ R$ and $R \circ S$ if $R, S \subseteq \mathbb{R} \times \mathbb{R}$ and:
   a) $xRy \iff 4x = y^2 + 6, \quad xSy \iff x - 1 = y$;
   b) $xRy \iff x = 2y, \quad xSy \iff y = x^3$;
   c) $xRy \iff \frac{1}{x} = y^2, \quad xSy \iff \sqrt{x-2} = 3y$;
   d) $xRy \iff (x - 3)^2 = y, \quad xSy \iff x = y^2 \wedge 2y = -x$.

6. Which of the following relations are
   • reflexive, • irreflexive, • symmetric, • antisymmetric,
   • strictly antisymmetric, • transitive, • dichotomous, • trichotomous:
   a) $\leq_{\mathbb{R}}$; b) $<_{\mathbb{R}}$; c) $\subseteq$; d) $\subset$; e) $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+,\ xRy \iff x \mid y$;
   f) ,,have a common point” on {planar circles}; g) ,,x knows y” on {people};
   h) ,,relative” on {people}; i) ,,sibling” on {people};
   j) $X = \{a, b, c\},\ R = \{(a, b), (b, a), (b, b), (a, c)\} \subseteq X \times X$; k) $X = \{1, 2, 3\},\ R = X \times X$;
   l) $X = \{1, 2, 3\},\ R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\} \subseteq X \times X$?

7. Find a relation on $\{1, 2, 3, 4\}$ which is:
   a) reflexive and not irreflexive; b) neither reflexive nor irreflexive;

c) antisymmetric and not symmetric; d) symmetric and not antisymmetric;
e) neither symmetric nor antisymmetric; f) both symmetric and antisymmetric;
g) both reflexive and trichotomous;
h) not reflexive, not transitive, not symmetric, not antisymmetric and not trichotomous.

8. Prove that $R \subseteq X \times X$ is an equivalence relation. Find the partition it determines.
a) $X = \{1,2,3,4,5\}$, $R = \{(1,1),(1,5),(2,2),(3,3),(3,4),(4,3),(4,4),(5,1),(5,5)\}$;
b) $X = \mathbb{Z}$, $aRb \iff 2 \mid (a+b)$; c) $X = \mathbb{N}$, $aRb \iff 3 \mid (a^2 - b^2)$;
d) $X = \mathbb{R}$, $aRb \iff a - b \in \mathbb{Q}$; e) $X = \mathbb{Z}$, $aRb \iff 5 \mid (a-b)$;
f) $X = \mathbb{Z} \times \mathbb{Z}$, $(a,b)R(c,d) \iff a + d = b + c$.
g) $X = \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a,b)R(c,d) \iff ad = bc$.

9. Find the equivalence relation on $X = \{a,b,c,d,e,f\}$ which determines the following partition:
a) $\{\{a,b,f\},\{c\},\{d,e\}\}$; b) $\{\{a\},\{b\},\{c\},\{d\},\{e,f\}\}$.

10. Is $R \subseteq X \times X$ a partial order? Is it a total order?
a) $X = \{a,b,c,d\}$, $R = \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,b),(c,c),(d,b),(d,c),(d,d)\}$;
b) $X = \mathbb{R}$, $aRb \iff a \le b$; c) $X = \mathbb{R}$, $aRb \iff a < b$;
d) $X = \mathbb{Z}^+$, $aRb \iff a \mid b$; e) $X = 2^{\mathbb{N}}$, $aRb \iff a \subseteq b$;
f) $X = \mathbb{Z}$, $aRb \iff |a| \le |b|$; g) $X = \mathbb{N} \times \mathbb{N}$, $(a,b)R(c,d) \iff a \le c \wedge b \le d$;
h) $X = \mathbb{R}[x]$ (set of real polynomials), $aRb \iff \deg a \le \deg b$.

11. Calculate the domain and the range, and decide whether it is a function:
a) $\{(x,y) \in \mathbb{R}^2 \mid 3 < x < 6 \wedge x < y < 2x\}$; b) $\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \le 1\}$;
c) $\{(x,y) \in \mathbb{R}^2 \mid y = (x-1)/(1-x^2)\}$; d) $\{(x,y) \in \mathbb{R}^2 \mid y(1-x^2) = x - 1\}$;
e) $\{(x,y) \in \mathbb{R}^2 \mid |x| = |y|\}$; f) $\{(x,y) \in \mathbb{R}^2 \mid y = x - \lfloor x \rfloor\}$.

12. Is $f \subseteq A \times B$ a function? If so, is it:
• injective, • surjective, • bijective?
a) $A = \{1,2,3,4,5\}$, $B = \{10,11,12\}$, $f = \{(1,11),(2,11),(4,12),(5,10)\}$;
b) $A = \{1,2,3,4\}$, $B = \{a,b,c,d,e\}$, $f = \{(1,a),(2,c),(3,d),(3,e),(4,a)\}$;
c) $A = \{1,2,3,4,5\}$, $B = \{a,b,c,d,e\}$, $f = \{(1,a),(4,e),(5,d)\}$;
d) $A = \{1,2,3\}$, $B = \{1,3,5\}$, $f = \{(1,1),(2,5),(3,5)\}$.

13. Which function is
• injective, • surjective, • bijective:
a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$; b) $f : \mathbb{R} \to \mathbb{R}_0^+$, $f(x) = x^2$; c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$;
d) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$; e) $f : \{a,b,c\} \to \{a,b,c\}$, $f(a) = b, f(b) = a, f(c) = c$?

14. Write a program that takes two finite binary relations R and S, and calculates $R \circ S$ and $S \circ R$.

15. How many
• reflexive, • irreflexive, • symmetric, • antisymmetric
relations exist on a given n-element set?

16. Prove that the inverse of a partial order is also a partial order.