Graph theory

Discrete mathematics I – exercises

1. Consider the following graph: $G = (V, E, \varphi)$, $V = \{A, B, C, D\}$, $E = \{e_1, e_2, e_3, e_4\}$, $\varphi = \{(e_1, \{A, B\}), (e_2, \{B, C\}), (e_3, \{A, C\}), (e_4, \{C, D\})\}$.
   a) Draw it. b) Determine $d(A)$, $d(B)$, $d(C)$ and $d(D)$.
   c) Draw $\overline{G}$. d) Are G and $\overline{G}$ isomorphic?

2. Draw all a) 3-vertex, b) 4-vertex, c) 5-vertex
   simple graphs up to isomorphy. How many connected, and how many regular graphs are there among them?

3. How many at most 5-vertex graphs are there which are isomorphic to their complement?

4. Is there any 9-vertex graph (not neccessarily simple) with the following degree sequence:
   a) $7, 7, 7, 6, 6, 6, 5, 5, 5$; b) $6, 6, 5, 4, 4, 3, 2, 2, 1$?

5. Is there any *simple* graph with the following degree sequence:
   a) $0, 2, 3, 3, 3, 3, 6$; b) $1, 3, 3, 4, 5, 6, 6$; c) $6, 6, 6, 6, 3, 3, 2, 2$; d) $3, 3, 3, 2, 2, 2, 1, 1, 1$;
   e) $7, 7, 7, 6, 6, 6, 5, 5, 5$; f) $2, 2, 3, 5, 6, 6, 6, 8, 8$; g) $6, 6, 5, 4, 4, 3, 2, 2, 1$?

6. Prove that the number of odd-degree vertices in a finite graph is always even.

7. A group of people does several handshakes. Prove that there are two people who shook hands with the same number of people.

8. Prove that if a connected graph with at least two vertices has fewer edges than vertices, then it must have a 1-degree vertex.

9. a) Prove that if in an at most $(2n + 1)$-vertex graph, all degrees are at least n, then the graph is connected.
   b) Is the statement true if the degree $n - 1$ is also allowed?

10. a) Prove that for any 6-vertex graph, either it or its complement contains K_3.
    b) Is it true for a 5-vertex graph?

11. Prove that a graph whose all degrees are at least 2 contains a cycle.

12. If a simple finite graph is not connected, is its complement neccessarily connected?