Complex numbers

Discrete mathematics I – exercises

1. Calculate, and give the result in algebraic form:
   a) $(3+2i)(2+i)$; b) $(2-5i)^2$; c) $\dfrac{1}{1+2i}$; d) $\dfrac{3+4i}{1-2i}$; e) $\dfrac{i}{(1-i)(1-2i)(1+2i)}$.

2. Solve for $x, y \in \mathbb{R}$:
   a) $(x+yi)(2-i) = x+3i$; b) $(x+i)(1+yi) = 3y+xi$; c) $(1+2i)x+(3-5i)y = 1-3i$;
   d) $\dfrac{5}{x+yi} + \dfrac{2}{1+3i} = 1$.

3. Solve for $z \in \mathbb{C}$:
   a) $\dfrac{z+i-3i\overline{z}}{z-4} = i-1$; b) $\left|\dfrac{z-3}{2-\overline{z}}\right| = 1 \ \wedge\ \operatorname{Re}\left(\dfrac{z}{2+i}\right) = 2$.

4. Convert to polar form:
   a) $1+i$; b) $\sqrt{3}-i$; c) $4i$; d) -3; e) $\dfrac{10}{\sqrt{3}-i}$; f) $\dfrac{2+3i}{5+i}$; g) $3-4i$; h) $-2+i$;
   i) $\dfrac{1+\sqrt{3}i}{2}$; j) $\dfrac{-1+\sqrt{3}i}{2}$; k) $\cos\dfrac{2\pi}{3} - i\sin\dfrac{2\pi}{3}$; l) $-\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$.

5. Calculate, using polar form:
   a) $(1+i)(-2+2i)$; b) $\dfrac{\sqrt{3}-i}{\sqrt{3}+i}$; c) $\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}i}{2}\right)^5$;
   d) $\dfrac{(1+i)^9}{(1-i)^7}$; e) $\left(\dfrac{2-2i}{\sqrt{3}-i}\right)^{24}$; f) $\left(\dfrac{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}\right)^{12}$.

6. Calculate the square root of:
   a) -4; b) $2i$; c) $-2+2\sqrt{3}i$; d) $3-4i$; e) $-7-24i$; f) $8+6i$.

7. Solve over $\mathbb{C}$:
   a) $x^3 = 1$; b) $x^3 = i$; c) $x^3 = 2+2i$; d) $x^4 = i$; e) $x^8 = \sqrt{3}-i$; f) $x^6 = 1+i$.

8. Calculate:
   a) the 6th roots of 64; b) the 5th roots of $-16\sqrt{3}+16i$; c) the 4th roots of $\dfrac{7+3i}{5-2i}$;
   d) the 6th roots of $\dfrac{1-i}{\sqrt{3}+i}$; e) the 4th roots of $\dfrac{-4}{(2+i)^3}$.

9. Draw the following complex numbers on the Gaussian plane:
   a) $\operatorname{Re}(z) \geq 2$; b) $\operatorname{Im}(z) \leq 1$; c) $\operatorname{Re}(z+2i) \leq 0$; d) $\operatorname{Re}(z+1) \geq \operatorname{Im}(z-2i)$;
   e) $|z| = 1$; f) $2 < |z| \leq 3$; g) $|z-1| \leq 2$; h) $|z-i-1| \leq 3$;
   i) $|z-1| < 1 \ \wedge\ \operatorname{Im}(z) > 0$; j) $|z+i| > 2 \ \wedge\ \operatorname{Re}(z) < 2$;

k) $|z+i| = |z-3i|$; l) $|z-3+2i| = |z+4-i|$;
m) $z = 1/\overline{z}$; n) $z + \overline{z} = 0$; o) $|z| = iz$; p) $z^3 = |z|$.

10. What geometric transformation does this function define on the Gaussian plane?
a) $z \mapsto 3z+2$, b) $z \mapsto (1+i)z$, c) $z \mapsto 1/\overline{z}$.

11. Let $z, w \in \mathbb{C}$ be two different points on the Gaussian plane.
a) Find the midpoint of the line segment between z and w.
b) Find the third vertex of the equilateral triangle whose two vertices are z and w.

12. List:
a) the primitive third roots of unity; b) the primitive fourth roots of unity;
c) the primitive second roots of unity; d) the primitive first roots of unity;
e) the primitive sixth roots of unity; f) the primitive eighth roots of unity.

13. Decide whether it is a complex root of unity. If so, find its order, and find those n for which it is an n^{th} root of unity / primitive n^{th} root of unity.
a) 1; b) -1; c) i; d) $1+i$; e) $\dfrac{1+i}{\sqrt{2}}$; f) $\dfrac{1+\sqrt{3}i}{2}$; g) $\dfrac{-1+\sqrt{3}i}{2}$;
h) $\cos\left(\dfrac{\pi}{361}\right) + i\sin\left(\dfrac{\pi}{361}\right)$; i) $\cos\left(\sqrt{2}\pi\right) + i\sin\left(\sqrt{2}\pi\right)$.

14. Prove that $\forall \varepsilon \in \mathbb{C} : \varepsilon^4 = i \implies 4 \mid o(\varepsilon)$.

15. Solve it using Cardano's formula (see e.g. Wikipedia):
a) $x^3 - 7x + 6 = 0$; b) $x^3 - 13x - 12 = 0$.

16. Prove that the multiplication of complex numbers is distributive over addition, i.e.:
$\forall x, y, z \in \mathbb{C} : x(y+z) = xy + xz$.

17. If $o(\varepsilon) = 128$, what can $o(i\varepsilon)$ be?