# Sets

**1.**
Let $U = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set, $A = \{1,2,3,4\}$, $B = \{0,2,4,8\}$ and $C = \{2,3,5,7\}$.

(a) Write down the following sets explicitly, i.e. by listing all their elements:

$A \cap B$ $\quad B \cup C$ $\quad A \setminus C$ $\quad \overline{C}$

(b) Consider the systems of sets $X = \{A, B, C\}$ and $Y = \{\{0,2,4,6,8\},\{1,3,5,7,9\}\}$.
Find the following sets:

$\cap X$ $\quad \cup X$ $\quad \cup Y$ $\quad \cap Y$

(c) Determine the truth value of each of the following statements:

| | | | |
|---|---|---|---|
| $4 \in B$ | $A \subseteq B$ | $\{\emptyset\} \subseteq \cup X$ | $3 \in A \cap B$ |
| $\{1,2\} \subseteq A$ | $A \in \cup Y$ | $A \subseteq \cup Y$ | $C \cap \emptyset = \emptyset$ |
| $2 \subseteq A$ | $\{2\} \subseteq A$ | $2 \in \cup X$ | $\{2\} \in \cap X$ |

**2.**
Let $\mathcal{A} = \{\{a,b,c\},\{a,d,e\},\{a,f\}\}$. Find the sets $\cup\mathcal{A}$ and $\cap\mathcal{A}$.

**3.**
Consider the system of sets $X = \{\{1,2,3\},\{2,3,4,5\},\{0,2,3,7\}\}$. Find the following sets: $\cap X$, $X \cup \{5,6,7,8\}$, $X \cup \{\{3,5,7\},\{1\},\{2\}\}$, $\cup(X \cup \{\{3,5,7\},\{1\},\{2\}\})$, $\cap(X \cup \{\{3,5,7\},\{1\},\{2\}\})$.

**4.**
Find the sets $A, B, C$, given that they satisfy the following:
$A \setminus B = \{1,3,5\}$, $A \cup B \cup C = \{1,2,3,4,5,6\}$, $(A \cap C) \cup (B \cap C) = \emptyset$, $C \setminus B = \{2,4\}$ and $(A \cap B) \setminus C = \{6\}$.

**5.**
Prove that the following equalities are true for any universal set $U$ and sets $A, B, C \subseteq U$ (hence these equalities are identities):

(a) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

(c) $A \cap B = B \cap A$

(d) $(A \cap B) \cap C = A \cap (B \cap C)$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(h) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(i) $A \cup \overline{A} = U$

(j) $A \cap \overline{A} = \emptyset$

(k) $\overline{\overline{A}} = A$

**6.**
Give an example for sets $A, B, C$ that satisfy all the conditions below:

$$A \cap B \neq \emptyset, \quad A \cap C = \emptyset, \quad (A \cap B) \setminus C = \emptyset.$$

**7.**
Prove that for any nonempty sets $A$ and $B$ the following equalities hold:
(a) $(A \setminus B) \cap B = \emptyset$
(b) $\big(A \cup \overline{B}\big) \cap \big(\overline{A} \cup \overline{B}\big) = \overline{B}$

**8.**
Let $A = \{a, b, c, d\}$, $B = \{c, d\}$ and $C = \{a, c, e\}$. Show that then $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. Is this statement true for all sets $A, B, C$?

**9.**
Show that the following statements are true for all sets $A, B, C$:
(a) if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$
(b) if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$
(c) $A \cup (B \cap A) = A$

**10.**
Write the following expression in its simplest possible form: $(A \cup (A \cap B) \cup (A \cap B \cap C)) \cap (A \cup B \cup C)$.

**11.**
Prove that the following equalities hold for any universal set $U$ and any sets $A, B, C \subseteq U$ (hence these equalities are identities).
(a) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
(b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(c) $A \setminus\ (A \setminus (B \setminus C)) = A \cap B \cap \overline{C}$

**12.**
Prove the following identity: $\overline{(\overline{A \cap B} \cup C) \cap \overline{A}} \cup \overline{B} \cup \overline{C} = A \cup \overline{B} \cup \overline{C}$.

**13.**
Decide which of the following statements are true *for all* sets $A, B, C$. Prove your statements.
(a) $\overline{A} \cap B = B \setminus A$
(b) $(A \cap B) \setminus C = (A \setminus B) \cap C$
(c) $(A \cup B) \cap (B \setminus A) = (A \cup B) \setminus (A \setminus B)$
(d) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
(e) $(A \cup B) \setminus A = B$
(f) $(A \cup B) \setminus C = A \cup (B \setminus C)$

**14.**
Prove the following identities.
(a) $A \triangle \emptyset = A$
(b) $A \triangle A = \emptyset$
* (c). $A \triangle (B \triangle C) = (A \triangle B) \triangle C$
* (d). $A \triangle (A \triangle B) = B$

**15.**
Prove that for any sets $A$ and $B$ we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, where $\mathcal{P}(A)$ denotes the power set of $A$. What can we say about the truth value of the statement obtained by replacing $\cap$ by $\cup$?

**16.**
Let $A, B, C, D$ be nonempty sets. Prove that then $A \times B \subseteq C \times D$ holds if and only if $A \subseteq C$ and $B \subseteq D$.

**17.**
Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:
$A \times A$, $A \times B$, $A \times A \times B$, $B \times A$, $(A \times A) \times B$, $A \times (A \times B)$, $A \triangle B$, $A \triangle C$.

**18.**
Prove that for any nonempty sets $A, B, C$ the following is true:
$(A \cup B) \times C = (A \times C) \cup (B \times C)$.