Relations

1.
Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$. Consider the following binary relation $R \subseteq A \times B$: $R = \{(1, 5), (1, 6), (1, 7), (3, 6), (3, 9), (4, 5), (4, 7), (4, 9)\}$.
(a) Find the domain and the range of R.
(b) Represent R on an arrow diagram.
(c) Let $H_1 = \{1, 2, 3\}$ and $H_2 = \{4\}$. Determine the restrictions $R|_{H_1}$ and $R|_{H_2}$ of R to sets H_1 and H_2, respectively.
(d) Find the inverse R^{-1} of R.

2.
Define $R \subseteq \mathbb{Z} \times \mathbb{Z}$ as $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 2b\}$. Determine the domain, range and inverse of R.

3.
Determine the image and the inverse image of the set $\{0\}$ under the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y^2 = 2 - x - x^2\}$. Describe those subsets A of $\mathbb{R}$ for which $R(A)$ contains only one element. Describe those subsets A of $\mathbb{R}$ for which $R^{-1}(A)$ contains only one element.

4.
Let $X = \{1, 2, 3\}$. In each of the following examples below decide if the relation R on X is reflexive, symmetric, anti-symmetric and/or transitive.
(a) $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
(b) $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$
(c) $R = \{(1, 2), (1, 3), (2, 1), (3, 1)\}$
(d) $R = \{(1, 2), (2, 3), (3, 1)\}$
(e) $R = \{(1, 2)\}$
(f) $R = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$
(g) $R = \{(1, 1), (2, 2), (2, 3), (3, 3)\}$
(h) $R = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$

5.
(a) Can a relation be both symmetric and anti-symmetric at the same time? Can a relation be both reflexive and irreflexive? Justify your answers.
(b) Prove that if a relation is both symmetric and anti-symmetric then it is also transitive.
(c) Prove that if a relation that is not the empty set is both irreflexive and symmetric, then it is not transitive.

6.
In each of the following examples, decide if the given relation is reflexive, irreflexive, symmetric, anti-symmetric and/or transitive, and find the domain and the range of the relation.
(a) $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \cdot b \text{ is odd}\}$
(b) $S = \{(a, b) \in B \times B \mid \text{ the surname of a is shorter than the surname of } b\}$ where B is the set of all Dimat 1. students at ELTE.

(c) $T_X = \{(A, B) \in P(X) \times P(X) \mid A \cap B \neq \emptyset\}$ where X is a given set.
(d) $U = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \gcd(a, b) > 1\}$
(e) $V = \{(x, y) \in K \times K| \mid x$ touches y from inside$\}$, where K is the set of all circles in a given plane.

7.
Consider the following $R \subseteq X \times X$ relation.
(a) $X = \{1, 2, 3, 4, 5\}$, $R = \{(1, 1), (1, 5), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 1), (5, 5)\}$
(b) $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $R = \{(1, 1), (1, 5), (1, 6), (1, 8), (2, 2), (2, 4), (3, 3), (3, 7), (4, 2), (4, 4), (5, 1), (5, 5), (5, 6), (5, 8), (6, 1), (6, 5), (6, 6), (6, 8), (7, 3), (7, 7), (8, 1), (8, 5), (8, 6), (8, 8)\}$

(1) Prove that R is an equivalence relation on A.
(2) Write down the partition of A induced by the equivalence relation R (in other words: find the quotient set A/R).

8.
In each example below find the equivalence relation on $\{a, b, c, d, e, f\}$ which corresponds to the given partition:
(a) $\{\{a, b, f\}, \{c\}, \{d, e\}\}$
(b) $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e, f\}\}$

9.
In each of the following examples prove that R is an equivalence relation (on the set which R is defined on in the example), and find the equivalence classes of R.
(a) $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + n \text{ is even}\}$
(b) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 \text{ is even}\}$
(c) $R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a - b \text{ is rational}\}$
(d) $R = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m^2 - n^2 \text{ is divisible by } 3\}$
(e) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 + y_1 = x_2 + y_2\}$
(f) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \cdot y_1 = x_2 \cdot y_2\}$

10.
Let $f \subseteq A \times A$ be a binary relation. Prove that $f = f^{-1}$ is true if and only if $f \subseteq f^{-1}$ holds.

11.
In each question below give an example for a relation on the set $\{1, 2, 3, 4\}$ satisfying the properties:
(a) reflexive and not irreflexive;
(b) anti-symmetric and not symmetric;
(c) symmetric and not anti-symmetric;
(d) both symmetric and anti-symmetric;
(e) neither symmetric nor anti-symmetric;
(f) both reflexive and trichotomous;
(g) not reflexive, not transitive, not symmetric, not anti-symmetric, and not trichotomous

12.
Let $A = \{1, 2, 3\}, B = \{a, b, c, d, e, f\}, C = \{2, 4, 6, 8\}$ and define $R \subseteq A \times B$ and $S \subseteq B \times C$ as follows: $R = \{(1, a), (1, b), (2, c), (2, f), (3, d), (3, e), (3, f)\}$ and $S = \{(a, 2), (a, 4), (c, 6), (c, 8),$

$(d,2),(d,4),(d,6),(f,8)\}$. Find the composition $S \circ R$.

13.
Let $A = \{1,2,3,4,5,6,7,8\}$ and $S, R \subseteq A \times A$. In each of the following cases determine the composition $S \circ R$.
(a) $R = \{(1,2),(1,3),(2,2),(3,3),(3,4),(4,1)\}$ and $S = \{(1,6),(2,3),(2,4),(3,1)\}$
(b) $R = \{(1,3),(1,4),(2,2),(2,4),(3,5),(5,6),(6,7)\}$ and $S = \{(1,2),(1,4),(2,3),(3,1),(3,2),(4,2),(4,6),(5,6),(7,2)\}$
(c) $R = \{(2,2),(2,4),(3,1),(3,4),(4,4),(5,3)\}$ and $S = \{(2,6),(3,7),(5,1),(5,6),(5,8),(6,2),(7,7)\}$
(d) $R = \{(6,1),(6,2),(7,3),(8,7)\}$ and $S = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),(5,1),(5,3),(5,5),(7,1),(7,2)\}$
Is the composition of relations a commutative operation? Hint: determine for example the composition $R \circ S$ in case (a).

14.
Let $R, S \subseteq A \times A$ be symmetric relations. Prove that $R \circ S$ is symmetric if and only if $R \circ S = S \circ R$.

15.
Let $R, S \subseteq \mathbb{R} \times \mathbb{R}$. In each of the following cases determine the compositions $S \circ R$ and $R \circ S$.
(a) $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid 4x = y^2 + 6\}$ and $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x - 1 = y\}$
(b) $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x = 2y\}$ and $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3\}$
(c) $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid \frac{1}{x} = y^2\}$ and $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid \sqrt{x-2} = 3y\}$
(d) $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - 6x + 5 = y\}$ and $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y \wedge 2y = x\}$

16.
Consider the following relations:
$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid |x-y| \le 3\}$, $\varphi = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 6x - 1 = 4y + 5\}$,
$\lambda = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 4 \mid 2x + 3y\}$, $\alpha = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 1.5x - 1.5 \le y\}$
Determine the compositions below.

$R \circ \varphi$ $\qquad$ $\varphi \circ \lambda$ $\qquad$ φ^3 $\qquad$ $\alpha \circ R$ $\qquad$ $R \circ \alpha$

17.
Let $A = \{2,3,6,8,9,12,18\}$, $R \subseteq A \times A$ and $aRb \iff a \mid b$.
(a) Prove that relation R is a partial order on set A.
(b) Draw the Hasse-diagram of R.

18.
(a) Prove that $\le$ is a partial order on $\mathbb{N}$, where the definition of $\le$ is:
$n, m \in \mathbb{N}, n \le m \iff \exists k \in \mathbb{N} : n + k = m$
(b) Prove that on $\mathbb{N} \times \mathbb{N}$, the following relation is a partial order:
$(m_1, n_1) R (m_2, n_2) \iff m_1 \le m_2 \wedge n_1 \le n_2$.

19.

Decide which of the following relations are partial order on the given set.

(a) P is the set of polynomials with real coefficients, $R \subseteq P \times P, fRg \iff \deg f \leq \deg g$

(b) $R \subseteq \mathbb{Z} \times \mathbb{Z}, aRb \iff |a| \leq |b|$

(c) V is the set of 10-unit-long vectors in $\mathbb{R}^2$, $R \subseteq V \times V, xRy \iff$ the polar angle of x is less than or equal to the polar angle of y (the polar angle is between $[0; 2\pi[$)

(d) $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2, xRy \iff$ the length of x is less than or equal to the length of y

20.

Decide which of the following relations are total orders on set $A = \{1, 2, 3, 4\}$.

(a) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

(b) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(c) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,4)\}$