Graph theory

Discrete mathematics 1. exercises

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1. Consider the following graph: $G = (V, E, \varphi)$, $V = \{A, B, C, D\}$, $E = \{e_1, e_2, e_3, e_4\}$, $\varphi = \{(e_1, \{A, B\}), (e_2, \{B, C\}), (e_3, \{A, C\}), (e_4, \{C, D\})\}$.
   a) Draw it. b) Determine $d(A)$, $d(B)$, $d(C)$ and $d(D)$.
   c) Draw $\overline{G}$. d) Are G and $\overline{G}$ isomorphic?

2. Draw all
   a) 3-vertex, b) 4-vertex, c) 5-vertex
   simple graphs up to isomorphy. How many connected, and how many regular graphs are there among them?

3. How many at most 5-vertex graphs are there which are isomorphic to their complement?

4. Prove that the number of odd-degree vertices in a finite graph is always even.

5. Is there any 9-vertex graph (not neccessarily simple) with the following degree sequence:
   a) $7, 7, 7, 6, 6, 6, 5, 5, 5$; b) $6, 6, 5, 4, 4, 3, 2, 2, 1$?

6. Is there any simple graph with the following degree sequence:
   a) $1, 3, 3, 4, 5, 6, 6$; b) $6, 6, 6, 6, 3, 3, 2, 2$; c) $3, 3, 3, 2, 2, 2, 1, 1, 1$;
   d) $7, 7, 7, 6, 6, 6, 5, 5, 5$; e) $2, 2, 3, 5, 6, 6, 6, 8, 8$; f) $6, 6, 5, 4, 4, 3, 2, 2, 1$?

7. A group of people makes several handshakes. Prove that there are two people who shook hand with the same number of people.

8. Prove that if a connected graph has less edges than vertices, then it must have a 1-degree vertex.

9. a) Prove that if in an at most $(2n + 1)$-vertex graph, all degrees are at least n, then the graph is connected.
   b) Is the statement true if the degree $n - 1$ is also allowed?

10. a) Prove that for any 6-vertex graph, either it or its complement contains K_3.
    b) Is it true for a 5-vertex graph?

11. a) Prove that if there is a walk between two vertices in a graph, then there is also a path between them.
    b) Prove that if there is a path from A to B and from B to C, then there is also a path from A to C.

12. Prove that a graph whose all degrees are at least 2 contains a cycle.

13. If a simple finite graph is not connected, is its complement neccessarily connected?

14. Draw all
    a) 3-vertex, b) 4-vertex, c) 5-vertex, d) 6-vertex
    trees up to isometry.

15. Which trees are isomorphic to their complement?

16. How many 8-vertex trees have exactly two 3-degree vertices?

17. Is there any 10-vertex forest with the following degree sequence: $1, 1, 1, 2, 3, 3, 4, 4, 5, 6$?

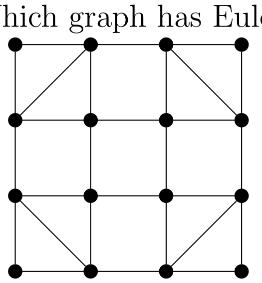
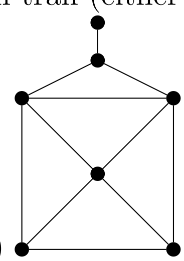
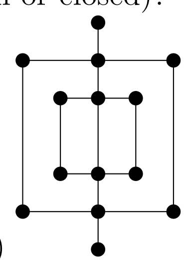
18. Prove that in any finite graph, the number of components and edges together is greater than or equal to the number of vertices.

19. Let G be a tree with vertex set V ($|V| \geq 2$), and let $a_k := \big|\{v \in V \mid d(v) = k\}\big|$. Prove that $a_1 \geq \sum_{k>2} a_k + 2$.

20. Prove that any two longest paths in a connected finite graph have at least one common vertex.

21. Prove that all longest paths in a finite tree go through one vertex.

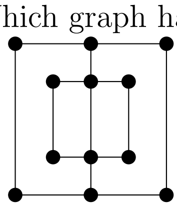
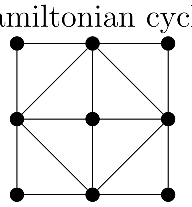
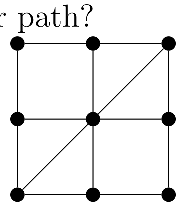
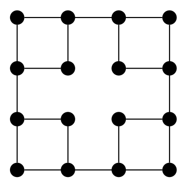
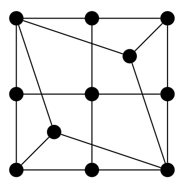
22. Which graph has Eulerian trail (either open or closed)?

a)  b)  c) 

23. Is it possible for a simple graph with even number of vertices and odd number of edges to have an Eulerian circuit?

24. Prove that the edges of a 4-regular graph can be coloured by red and blue such that all vertices have two incident red and two blue edges.

25. Which graph has Hamiltonian cycle or path?

a)  b)  c)  d)  e) 

26. Prove that if a bipartite graph has a Hamiltonian cycle, then the two vertex classes have the same size.

27. Is it possible for a knight to visit all squares of a 9×9 chessboard and then return to the initial position without touching any other square twice?

28. Prove that a graph with a Hamiltonian cycle remains connected if:
a) one edge is removed; b) one vertex is removed.

29. Prove that if a graph has k vertices which, when removed, make the graph fall into more than k components, then the graph has no Hamiltonian cycle.

30. Prove that for any $n \geq 5$, there exists an n-vertex graph such that:
a) both the graph and its complement contains a Hamiltonian cycle;
b) neither the graph nor its complement contains a Hamiltonian cycle.

31. Prove that 100 people can sit around a big round table at least 25 times if no two people sit next to each other more than once.

32. Prove that a cycle can be formed from a domino set (labeled from 0 to 6).

33. Prove that the Petersen graph contains no Hamiltonian cycle, but after removing any of its edges, the resulting graph contains one.