Complex numbers

1.
Calculate the following expressions in algebraic form.

$\sqrt{-16}$ $\quad$ $\sqrt{-25}$ $\quad$ $(2i)^2$ $\quad$ $2i + 5i$ $\quad$ $\dfrac{4i}{2i}$

2.
Let $z \in \mathbb{C}, z = -2 + 7i$. Determine the following:

$\operatorname{Re} z$ $\quad$ $\operatorname{Im} z$ $\quad$ $-z$ $\quad$ $\overline{z}$ $\quad$ $|z|$

3.
Calculate the value $\dfrac{4+3i}{(2-i)^2}$ in algebraic form.

4.
Solve the following equation on the set of complex numbers:

$$\frac{x + i - 3i\overline{x}}{x - 4} = i - 1.$$

5.
Find the complex number(s) $z \in \mathbb{C}$ satisfying the conditions:

$$\left|\frac{z-3}{2-\overline{z}}\right| = 1 \land \operatorname{Re}\left(\frac{z}{2+i}\right) = 2$$

6.
Let $z \in \mathbb{C}, z = 2 + 5i$. Find the absolute value and the argument of z. Represent z on the complex plane (also called Gaussian plane).

7.
Write the following complex numbers in polar form:

(a) $1 + i$

(b) $-\sqrt{3} + i$

(c) $\dfrac{9}{2} - \dfrac{9\sqrt{3}}{2}i$

(d) $-\dfrac{\sqrt{14}}{2} - \dfrac{\sqrt{14}}{2}i$

(e) $4i$

(f) i

(g) 10

8.
Calculate the following, using the polar form of complex numbers:

(a) $\left(\frac{9}{2}-\frac{9\sqrt{3}}{2}i\right)\left(-\frac{\sqrt{14}}{2}-\frac{\sqrt{14}}{2}i\right)$

(b) $\left(-\frac{3\sqrt{3}}{2}-\frac{3}{2}i\right)\left(\frac{\sqrt{3}}{3}+\frac{1}{3}i\right)$

(c) $\dfrac{-\frac{3\sqrt{3}}{2}-\frac{3}{2}i}{\frac{\sqrt{3}}{3}+\frac{1}{3}i}$

(d) $\left(\frac{5\sqrt{3}}{12}-\frac{5}{12}i\right)^{10}$

(e) $\left(-\frac{\sqrt{10}}{2}-\frac{\sqrt{10}}{2}i\right)^{15}$

(f) $\left(\frac{5}{2}-\frac{5\sqrt{3}}{2}i\right)^{23}$

(g) $(1+i)^8\cdot\left(5\sqrt{3}-5i\right)^3$

(h) $\left(\dfrac{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}\right)^{12}$

(i) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$

9.
Determine the complex roots below:
(a) 2^{nd} roots of -64;
(b) 3^{rd} roots of -64;
(c) 6^{th} roots of $1-\sqrt{3}i$;
(d) 5^{th} roots of $-7\sqrt{3}+7i$;
(e) 8^{th} roots of $-\frac{7}{2}+\frac{7}{2}i$;
(f) 2^{nd} roots of $-6\sqrt{3}+6i$;
(g) 7^{th} roots of $\dfrac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^8}{(1+i)^5}$;

10.

Using the polar form of complex numbers, calculate the value of $z=\dfrac{\left(2+2\sqrt{3}i\right)^{10}}{(-1+i)^{83}}$, giving your answer both in algebraic and in polar forms. Find all complex numbers w such that $w^3=z$, giving your answers in polar form.

11.

Express $z=\dfrac{(1+i)^8}{(1-\sqrt{3}i)^6}$ in algebraic form.

12.
In each of the following examples describe the geometric transformation of the Gaussian plane determined by the given function.
(a) $z \mapsto 3z + 2$
(b) $z \mapsto (1 + i)z$
(c) $z \mapsto 1/\overline{z}$

13.
In the Gaussian plane the center of a square is at point $C = 1 + 2i$ and one of its vertices is at $A = 5+4i$. Determine the complex numbers represented by the other three vertices of this square.

14.
Let $z \neq w$ be two complex numbers. Find the complex number represented by the midpoint of the line segment between z and w. Consider the two equilateral triangles having both z and w among their vertices. Find the complex number represented by the third vertex of each of these two equilateral triangles. Find the complex number represented by the centroid (or median point) of each triangle.

15.
Find the vector obtained by rotating the vector $\begin{bmatrix} 2 \\ -2\sqrt{3} \end{bmatrix} \in \mathbb{R}^2$ in the plane by each of the following angles: (a) $34°$ (b) $-176°$.

16.
Consider the sets

$$A = \{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$$
$$B = \{z \in \mathbb{C} \mid \operatorname{Im} z < 2\}$$
$$C = \{z \in \mathbb{C} \mid |z - 2| = 3\}$$
$$D = \{z \in \mathbb{C} \mid z^2 - (3 + 2i)z + (5 + 5i) = 0\}$$

Represent each of the following sets in the Gaussian plane:

(a) A (b) B (c) C (d) D (e) $A \cap B$ (f) $A \cup B$
(g) $A \cap C$ (h) $B \cup C$ (i) $A \setminus B$ (j) $A \triangle B$ (k) $A \cap D$ (l) $C \setminus \overline{B}$

17.
Represent each of the following sets in the Gaussian plane:
(a) $\{z \in \mathbb{C} \mid |z - i + 2| = 10\}$
(b) $\{z \in \mathbb{C} \mid \operatorname{Re} z = \operatorname{Im} z\}$
(c) $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \operatorname{Im} z\}$
(d) $\{z \in \mathbb{C} \mid |z - 2| \leq |z + 3|\}$
(e) $\{z \in \mathbb{C} \mid 2 < |z + i - 2| \leq 4\}$

18.
For each number z below, decide if z is a complex root of unity. If z is a complex root of unity then:

(1) find the order of z;

(2) find all those values $n \in \mathbb{N}^+$ for which z is an n^{th} root of unity and

(3) find those values $n \in \mathbb{N}^+$ for which z is a primitive n^{th} root of unity.

(a) 1
(b) -1
(c) i
(d) $1+i$
(e) $\frac{1+i}{\sqrt{2}}$
(f) $\frac{1+\sqrt{3}i}{2}$
(g) $\frac{-1+\sqrt{3}i}{2}$
(h) $\frac{-1+\sqrt{3}i}{2}$
(i) $\cos\left(\sqrt{2}\pi\right)+i\sin\left(\sqrt{2}\pi\right)$
(j) $\cos\left(\frac{\pi}{361}\right)+i\sin\left(\frac{\pi}{361}\right)$

19.
Show that if $\varepsilon^4 = i$ then $4 \mid o(\varepsilon)$.

20.
Suppose $o(\varepsilon) = 128$. What is $o(i \cdot \varepsilon)$? Justify your answer.

21.
(a) One of the fourth roots of $z = -1-\sqrt{3}i$ is $w_0 = \frac{\sqrt[4]{2}}{2}(\sqrt{3}-i)$. Using a primitive fourth root of unity, find all fourth roots of unity. With the help of these, calculate all fourth roots of z.
(b) One of the sixth roots of $z = -i$ is $w_0 = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. Using a primitive sixth root of unity, find all sixth roots of unity. With the help of these, calculate all sixth roots of z.
(In this question, the formula for calculating the n^{th} roots of a complex numbers is not allowed.)